Module Two: Differential Calculus(continued)

synopsis of results and problems

(student copy)

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1 Syllabus

- Taylor's and Maclaurin's theorems for function of one variable(statement only)- problems. Evaluation of Indeterminate forms.
- Partial derivatives Definition and simple problems, Euler's theorem problems, total derivatives, partial differentiation of composite functions, Jacobians-definition and problems.
- extreme values of functions of two variables .

2 Taylor's and maclaurin's expansions

1. Taylor's theorem

   Proposition 4.23 (Taylor's Theorem). *Let $n \in \mathbb{Z}$, $n \geq 0$, and $f : [a, b] \to \mathbb{R}$ be such that $f', f'', \ldots, f^{(n)}$ exist on $[a, b]$ and further, $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$ such that*

   $$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

2. Maclaurin series is a special case of taylor series. This occurs when $a = 0$.
3. L'hopital rule

Proposition 4.37 (L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Forms). *Let $c \in \mathbb{R}$ and $f, g : (c-r, c) \to \mathbb{R}$ be differentiable functions such that*

$$\lim_{x \to c^-} f(x) = 0 \quad \text{and} \quad \lim_{x \to c^-} g(x) = 0.$$

Suppose $g'(x) \neq 0$ for all $x \in (c-r, c)$, and

$$\frac{f'(x)}{g'(x)} \to \ell \text{ as } x \to c^-.$$

Then

$$\frac{f(x)}{g(x)} \to \ell \text{ as } x \to c^-.$$

Here ℓ can be a real number or ∞ or $-\infty$.

Proof. Extend f, g to $(c-r, c]$ by putting $f(c) = g(c) = 0$. Let (x_n) be a sequence in $(c-r, c)$ such that $x_n \to c$. Since $g'(x) \neq 0$ for all $x \in (c-r, c)$, by Cauchy's Mean Value Theorem we see that for each $n \in \mathbb{N}$,

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f'(c_n)}{g'(c_n)} \quad \text{for some } c_n \text{ between } x_n \text{ and } c.$$

Now $x_n \to c$ implies that $c_n \to c$, and hence the quotient above tends to ℓ as $n \to \infty$. Thus $f(x_n)/g(x_n) \to \ell$. □

Exercises:

(i)

$$\lim_{x \to 2} \frac{\sqrt{x^2+5} - 3}{x^2 - 4}$$

(ii)

$$\lim_{x \to 1} \frac{x^3 - 3x^2 - 3x + 1}{x^3 + x^2 - 5x + 3}$$

4. **Problems**

(i) Let $a_0, a_1, \ldots, a_n$ are real numbers such that

$$\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + \frac{a_0}{1} = 0$$

Then show that the polynomial function $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has at least one root in $(0, 1)$.

(ii) Show that if $0 < a < b$ then, $1 - \frac{a}{b} < \log(\frac{b}{a}) < 1 - \frac{b}{a}$

(iii) Find the 4th degree taylor polynomial approximating $\frac{1}{1+x}$

(iv) Find the maclaurin's series for $e^x, \sin x, \cos x, \tan x, \log(1+x), \log(1-x), \tan^{-1}(x), \sin h(x), \cos h(x)$

(v) Prove that

$$e^{e^x} = e\left(1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \dots\right)$$

(vi) Find the expansion of $\log(1 + x + x^2)$

(vii) Expand $(1 + x)^x$ upto 4th power of x.

(viii) Expand $\log(\sec x)$

(ix)

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = -\frac{e}{2}$$

(x)

$$\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$$

(xi)

$$\lim_{x\to 0} (\sin x)^x$$

(xii)

$$\lim_{x\to \pi/4} (\tan x)^{\tan 2x}$$

3 Partial differentiation

1. Some surfaces

   - $z(x, y) = x^2 + y^2$

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Plot3D[x^2 + y^2, {x, -3, 3}, {y, -3, 3}]
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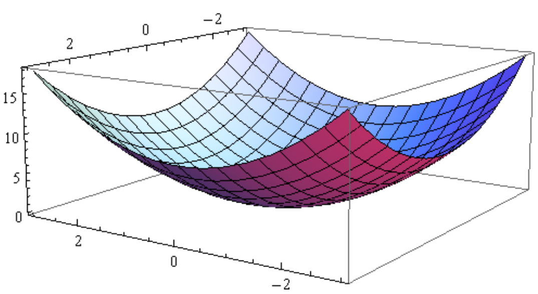

   - $z(x, y) = \sqrt{1 - (x^2 + y^2)}$

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Plot3D[Sqrt[1 - (x^2 + y^2)], {x, -1, 1}, {y, -1, 1}]
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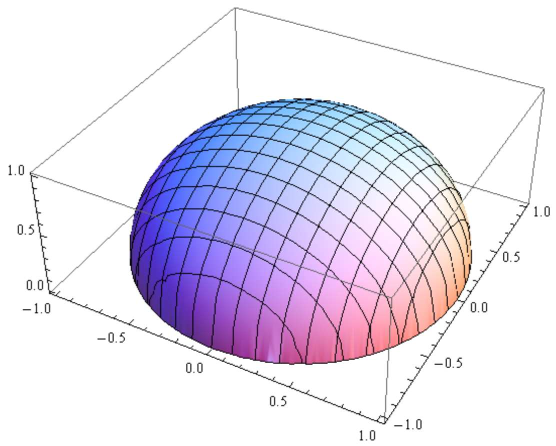

- $z(x, y) = \frac{x^2 y}{x^4 + y^2}$

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Plot3D[x^2 y/(x^4 + y^2), {x, -1, 1}, {y, -1, 1}]
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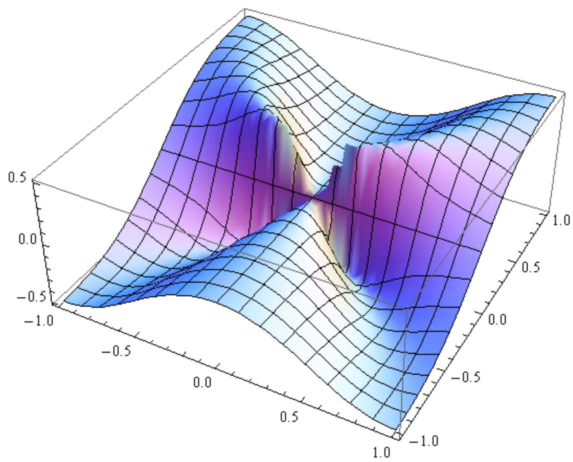

2. Continuity of a function at a point by double limits.

   For example, $f(x, y) = xy$ is continuous at $(0, 0)$. But, $g(x) = \frac{xy}{x^2+y^2}$ is not as we see that

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y/x}{1 + (y/x)^2} = \frac{m}{1 + m^2}$$

   as the limit depends on the slope of the line we approach the point $(0, 0)$. The picture below gives us the intuition.

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Plot3D[x y, {x, -1, 1}, {y, -1, 1}]
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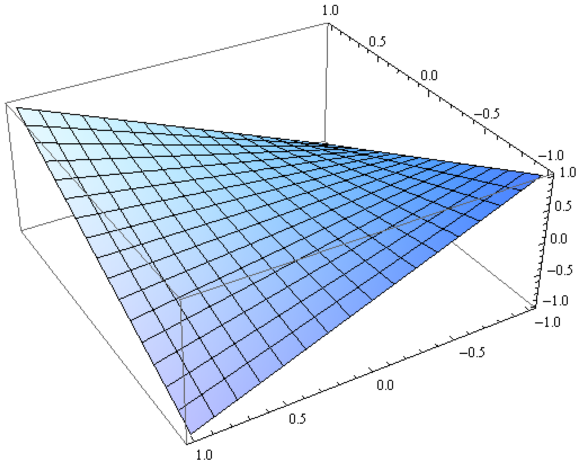
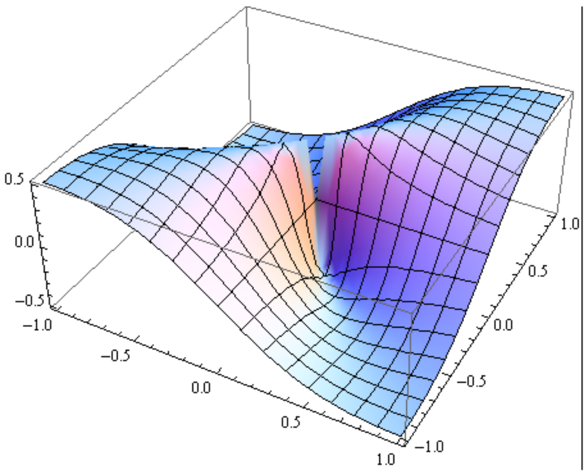


$$\texttt{Plot3D}\left[\frac{x\,y}{x^2+y^2},\ \{x, -1, 1\},\ \{y, -1, 1\}\right]$$



(a) $f(x,y) = \frac{x-y}{x+y}$. Prove that limit does not exists as $(x,y) \to (0,0)$

3. A function $f : D \to \mathbb{R}$ is said to be **differentiable** at (x_0, y_0) if there exists $(a,b) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x+0, y_0) - ah - bk}{\sqrt{h^2+k^2}} = 0$$

The partial derivatives are $f_x = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and $f_y = \lim_{k\to 0} \frac{f(x+k)-f(x)}{k}$.

(a) Provided $f(x,y) = x^3y + e^{xy^2}$, find $f_x, f_y, f_{xy}, f_{xx}, f_{yy}$

(b) Prove that $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ satisfies the equation $f_{xx} + f_{yy} + f_{zz} = 0$

4. If partial derivatives of $f(x,y)$ are continuous at (x_0, y_0), then f is continuous.

5. Existence of partial derivatives (at some point) does not assure that the function is differentiable at the point.

   Recall that

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \end{cases} \tag{1}$$

   is not continuous at $(0,0)$ although $f_x(0,0)$ and $f_y(0,0)$ exist.

6. Derivative or Total derivative of the function $f(x,y)$ is defined as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

   $f(x,y) = 0$ defines a surface and the RHS of the above equation gives the tangent plane when evaluated at some particular point (x_0, y_0).

7. Differentiation of composite function (chain rule for partial derivatives)

   Let $f(x,y)$ and $x(r,s)$ and $y(r,s)$. We express r and s as functions of x and y. Then we could use the chain rule as follows,
   $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x}$

   Exercises:

   (a) Let $U = z\sin(y/x)$ where $x = 3r^2 + 2s, y = 2r^2 - 3s^2$. Find U_r and U_s.

   (b) If $x = r\cos\theta$, $y = r\sin\theta$ and $V(x,y)$ be some function, show that

$$V_x^2 + V_y^2 = v_r^2 + \frac{v_\theta^2}{r^2}$$

8. Euler's theorem about a homogeneous function: Let $f(x,y)$ be a homogeneous function of degree p that is $f(ax, ay) = a^p f(x,y)$. Then, $xf_x + yf_y = pf$

   Exercise: If $f(x,y) = x^4y^2\sin^{-1}(y/x)$, prove that $xf_x + yf_y = 6F$

9. Problems

   (a) If $x^x y^y z^z = c$, prove that when $x = y = z$, we have

$$z_{xy} = \frac{-1}{x\log(ex)}$$

(b) If $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then

$$xu_x + yu_y = \frac{\sin 2u}{2}$$

(c) If u is a homogeneous function of degree n prove that

$$xu_{xx} + yu_{xy} = (n-1)u_x$$

and

$$xu_{xy} + yu_{yy} = (n-1)u_y$$

and hence

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)u$$

(d) If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, show that

$$xu_x + yu_y + zu_z = -3\tan u$$

(e) If $u = \log r$ where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, then

$$u_{xx} + u_{yy} + u_{zz} = \frac{1}{r^2}$$

(f) If $x = r\cos\theta$ and $y = r\sin\theta$, then prove that

$$x_r = r_x, x_\theta = r^2\theta_x$$

and hence prove that $\theta_{xx} + \theta_{yy} = 0$

(g) If $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$r_{xx} + r_{yy} = \frac{r_x^2 + r_y^2}{r}$$

(h) If $x = 2r - s$ and $y = r + 2s$, find u_{yx} in terms of derivatives of u with respect to r and s.

4 Jacobian and independence

1. **Differentiation in the implicit case**

When $F(x, y, z)$ is not expressed explicitly, we do the following:

$$0 = dF = F_x dx + F_y dy + F_z dz \qquad (2)$$

We assume that there exists a function $f(x,y)$ such that $z = f(x,y)$. Then,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{3}$$

On comparison of the above two equations, we find that $\dfrac{\partial f}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial f}{\partial y} = -\dfrac{F_y}{F_z}$.

2. Definition of jacobian

Definition 352 (2-D case) *Suppose that x and y are two independent variables which can be expressed in term of two other independent variables u and v by the formula $x = g(u,v)$ and $y = h(u,v)$.* ***The Jacobian*** *of x and y with respect to u and v, denoted $\dfrac{\partial (x,y)}{\partial (u,v)}$ or $J(u,v)$, is*

$$J(u,v) = \frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

3. An example

Example 354 *Recall, when switching from Cartesian to polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$. The Jacobian of x and y with respect to r and*

θ is

$$\begin{aligned} \frac{\partial (x,y)}{\partial (r,\theta)} &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ \frac{\partial (x,y)}{\partial (r,\theta)} &= r \end{aligned}$$

4. Some results about the Jacobian

Theorems on Jacobians

In the following we assume that all functions are continuously differentiable.

1. A necessary and sufficient condition that the equations $F(u, \upsilon, x, y, z) = 0$ and $G(u, \upsilon, x, y, z) = 0$ can be solved for u and υ (for example) is that $\frac{\partial(F, G)}{\partial(u, \upsilon)}$ is not identically zero in a region $\Re$.

   Similar results are valid for m equations in n variables, where $m < n$.

2. If x and y are functions of u and υ while u and υ are functions of and s, then (see Problem 6.43)

$$\frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(u, \upsilon)} \frac{\partial(u, \upsilon)}{\partial(r, s)} \tag{13}$$

   This is an example of a *chain rule* for Jacobians. These ideas are capable of generalization (see Problems 6.107 and 6.109, for example).

3. If $u = f(x, y)$ and $\upsilon = g(x, y)$, then a necessary and sufficient condition that a functional relation of the form $\phi(u, \upsilon) = 0$ exists between u and υ is that $\frac{\partial(u, \upsilon)}{\partial(x, y)}$ be identically zero Similar results. hold for n functions of n variables.

5. Problems

(a) If $u = x^3 y$ find u_t provided $x^5 + y = t$ and $x^2 + y^3 = t^2$

(b) If $u^2 - v = 3x + y$ and $u - 2v^2 = x - 2y$, find u_x, u_y, v_x, v_y

(c) If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}(x)$, are u and v functionally related?

(d) Let x and y be functions of u and v. Further, u and v be functions of r and s. Prove that

(i)

$$\frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(r, s)}$$

(ii)

$$\frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(x, y)} = 1$$

(e) If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, then show that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2uv(u - v)}$$

(f) If $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent unless

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$$

5 Extreme values of functions of two variable

1. Tests

Test 1

Let $D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$. Suppose that at some point (x_0, y_0) we have $\nabla f(x_0, y_0) = \langle 0, 0\rangle$.

$D(x_0, y_0) > 0 \Rightarrow f(x, y)$ has a local max or min at (x_0, y_0).

$D(x_0, y_0) < 0 \Rightarrow f(x, y)$ has a saddle at (x_0, y_0).

$D(x_0, y_0) = 0 \Rightarrow$ No information about $f(x_0, y_0)$.

Test 2

Let $D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$. Suppose that at some point (x_0, y_0) we have $\nabla f(x_0, y_0) = \langle 0, 0\rangle$ and $D(x_0, y_0) > 0$.

$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0 \Rightarrow f(x, y)$ has a local min at (x_0, y_0).

$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0 \Rightarrow f(x, y)$ has a local max at (x_0, y_0).

2. Find the maxima, minima and saddles of the functions: $xy + 2x - 3y - 6$ (ans. (3,2) is a saddle) and $x^3 - xy + y^2$
3. Discuss the extreme points of $\sin(x + y)$
4. If at some point (x_0, y_0) if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f_{yy}(x_0, y_0) > 0$
5. If the perimeter a triangle is constant, then the area is maximum when the triangle is equilateral.
6. If $x + y + z = 1, x > 0, y > 0, z > 0$, find the extreme values of $xy + yz + zx$

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