Module One: Problems

Student Copy

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Abstract

The results are given in terse form. Please refer some textbook for details about results. Given the time constraint, we may not be able to solve all the problems in the classroom. Best wishes.

1 Syllabus

Differential Calculus 1

- Primary purpose of this is to apply this when finding Taylor's series for a function.
- Determination of n th order derivatives of Standard functions - Problems. Leibnitzs theorem (without proof) - problems.
- Polar Curves - angle between the radius vector and tangent, angle between two curves, Pedal equation for polar curves. Derivative of arc length - Cartesian, Parametric and Polar forms (without proof) - problems. Curvature and Radius of Curvature Cartesian, Parametric, Polar and Pedal forms and problems

2 Finding the n th derivative

1. $D^n((ax+b)^m) = m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n}$. Break it into cases when $m < n, m > n$. When $m = -1$, differentiate once and then proceed.
2. $D^n(a^{mx}) = m^n a^{mx}(\log a)^n$

3. $D^n(\cos(ax+b)) = a^n \cos(ax+b+n\frac{\pi}{2})$ and a similar expression for sin
4. Product of exponential and trig. $D^n(e^{ax}\cos(bx+c)) = a^n \cos(bx+c+n\frac{\pi}{2})$ and a similar expression for sin
5. Note that the n th derivative operator is additive (indeed a linear transformation), but not multiplicative.
6. Leibnitz rule is an outcome of *chain rule of differentiation.* Let $u(x)$ and $v(x)$ be two n times differentiable function of x. We have

$$(u+v)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

where $f^{(k)}$ stands for k th derivative of f.

3 Problems

1. Find the n th derivative of $\frac{x}{(x-1)(x+1)}$

ans. $\frac{1}{2}\left(\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}}\right)$

2. $e^{ax}\cos^2 x \sin x$

ans. $\frac{1}{4}\left(r^n e^{ax}\sin(x+n\phi)\right) + \frac{1}{4}\left(r_1^n e^{ax}\sin(x+n\varphi)\right)$ where
$r = \sqrt{a^2+1}, \tan\phi = 1/a, r_1 = \sqrt{a^2+9}, tan\varphi = 3/a$

3. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

ans. $2(-1)^{n-1}(n-1)! r^{-n} \sin n\theta$ where $r = \sqrt{x^2+1}, \theta = \tan^{-1}(1/x)$

4. $y = x \log \frac{x-1}{x+1}$

ans. $(-1)^{n-2}(n-2)!\left(\frac{x-n}{(x-1^n)} - \frac{x+n}{(x+1^n)}\right)$

5. $x^2 \sin 3x$

ans. $3^n x^2 \sin\left(3x+n\frac{\pi}{2}\right) + 3^{n-2}n(n-1)\sin\left(3x+(n-1)\frac{\pi}{2}\right) + 3^{n-1}2nx\sin\left(3x+(n-2)\frac{\pi}{2}\right)$

6. If $y = (1-x)^{-a}e^{-ax}$, show that

$$(1-x)y^{(n+1)} - (n+ax)y^{(n)} - nay^{(n-1)} = 0$$

7. If $y = \cos(m \sin^{-1}(x))$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

8. $D^{n^n}(2n-1)^n$ (Find n^n the derivative of $(2n-1)^n$)

ans. 2^n

4 Polar Curves

1. Angle between radius vector and the tangent (in polar co-ordinates)

   We have $\tan(\psi) = \frac{dy}{dx}$. Hence,

$$\tan\psi = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d(r\sin\theta)}{d\theta}}{\frac{d(r\cos\theta)}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{1}$$

   We have, $\mu = \psi - \theta$, and it gives

$$\tan\mu = \frac{r}{\frac{dr}{d\theta}} \tag{2}$$

2. Pedal and pedal equation

   The pedal with respect to origin is the perpendicular distance from origin to the foot of perpendicular lying on the tangent and passing through origin. Hence, the length of the pedal p,

$$p = r\sin\mu \tag{3}$$

   The pedal equation is obtained by eliminating (2) and (3). If done analytically, this yields the following

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \tag{4}$$

3. Polar sub tangent and polar subnormal

   Polar Sub tangent is the length of the projection of the tangent on the line perpendicular to the radius vector. Definition of polar sub normal is the obvious. We have $p_t = r\tan\mu$ and $p_n = r\cot\mu$

   Using (2), we get

$$p_t = r^2 \frac{d\theta}{dr} \tag{5}$$

$$p_n = \frac{dr}{d\theta} \tag{6}$$

By Pythagoras theorem, we have

$$\text{Length of polar tangent} = r\sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} \tag{7}$$

$$\text{Length of polar normal} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \tag{8}$$

4. The angle (at a point of intersection) between two curves is the the difference of the slopes of the tangents at that point. Some two curves are said to be *orthogonal* at a point of intersection if their tangents make an angle of $\frac{\pi}{2}$ wrt to each other.

5 Problems

1. The curves $r^m = a^m \cos m\theta$ and $r^n = a^n \cos n\theta$ are orthogonal.
2. $r = a(1 - \cos\theta)$. Find angle between radius vector and the tangent etc.
3. If $r = ae^{m\theta^2}$ where a and m are constants. Prove that $\frac{p_n}{p_t} \sim \theta^2$
4. Find the point(s) and angle of intersection of $r = 3\cos\theta$ and $r = 1 + \cos\theta$ (ans. $\frac{\pi}{6}$)
5. Find the pedal and pedal equations for the following:
   (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ans. $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{(ab)^2}$)
   (b) $r^m = a^m \sin(m\theta)$ (ans: $pa^m = r^{m+1}$)

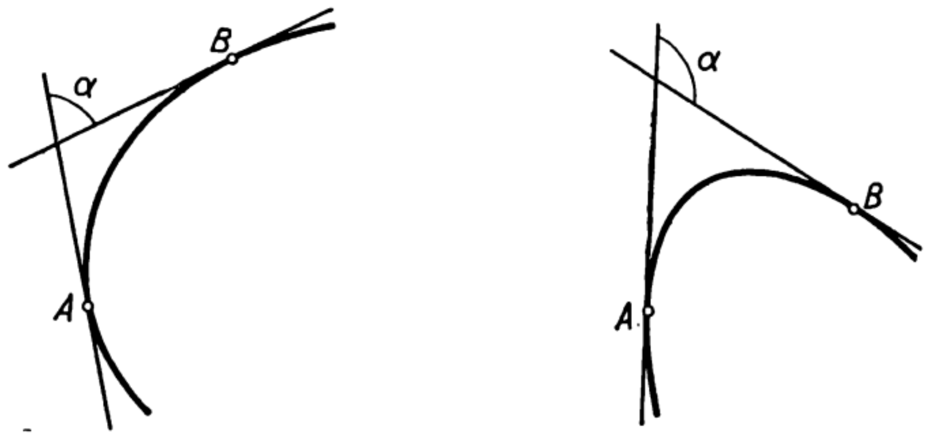
6. Show that the length of the polar tangent is constant in the curve

$$\theta = \cos^{-1}\left(\frac{r}{k}\right) - \sqrt{\frac{r^2 - k^2}{r^2}}$$

6 Arc length and Curvature

1. If s is the arc length, then $(ds)^2 = (dx)^2 + (dy)^2$. Further, s as a function of x gives us $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

2. Note that $\frac{ds}{dx} = \sqrt{1 + (\tan\psi)^2} = \sec\psi$. Further, $\cos\psi = dx/ds$ and $\sin\psi = dy/ds$



3. curvature of arc $AB = \frac{\text{angle of contingence}}{\text{length of the arc AB}}$. Note that circle has same curvature at every point and straight line's curvature is uniformly zero.

4. curvature= $\frac{\Delta\psi}{\Delta s}$. Then, take the limit to show that curvature= $\frac{d\psi}{ds}$

5. Use the fact that $\psi = \tan^{-1}\left(\frac{dy}{dx}\right)$ and prove that

$$K = \left| \frac{d\psi}{ds} \right| = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

6. Curvature in parametric form is obtained by applying the previous formula. Let $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$. Then,

$$K = \frac{| x'y'' - x''y' |}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$

7. curvature in polar form:

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

and

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}$$

8. Also, $\frac{ds}{dr} = \sqrt{1+\tan^2\mu} = \sec\mu$

9. curvature in polar form: (obtained by putting $x = r\cos\theta$ and $y = r\sin\theta$)

$$K = \frac{\mid r^2 + 2r_1^2 - rr_2 \mid}{(r^2 + r_1^2)^{3/2}}$$

where $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$

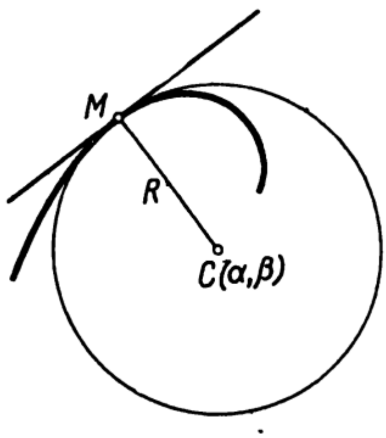
10. When the equation in implicit:

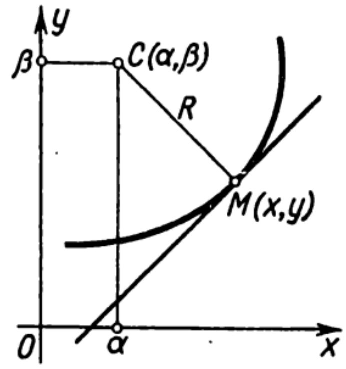
We have $\frac{dy}{dx} = -\frac{f_x}{f_y}$,

$$K = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(f_x^2 + f_y^2)^{3/2}}$$

11. Radius of curvature is $1/K$. Motivation comes from the circle case where $K = 1/r$

12. Draw a normal, at the point M, to a curve in the direction of the concavity of the curve, and lay off a segment MC equal to the radius R of the curvature of the curve at the point M. The point C is called the centre of curvature of the given curve at M; the circle, of radius R, with centre at C (passing through M) is called the circle of curvature of the given curve at the point M.





13. Centre of curvature of the is

$$\left(x - \frac{y'(1+(y')^2)}{y''}, y + \frac{1+(y')^2}{y''}\right)$$

and in parametric form we have

$$x(t) - \frac{y'((x')^2+(y')^2)}{(x'y''-x''y')}, y(t) + \frac{x'((x')^2+(y')^2)}{(x'y''-x''y')}$$

14. We can obtain radius of curvature via the pedal equation as $R = r\frac{dr}{dp}$

7 Problems

(a) Determine the curvature of the cycloid $x = a(t - \sin t), y = a(1 - \cos t)$ at an arbitrary point (x, y).

(b) Find the curvature:

1. $b^2x^2 + a^2y^2 = a^2b^2$ at the points $(0, b)$ and $(a, 0)$. *Ans.* $\frac{b}{a^2}$ at $(0, b)$; $\frac{a}{b^2}$ at $(a, 0)$.

(c) Lot of problems

5. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at an arbitrary point. *Ans.* $\frac{1}{3(axy)^{\frac{1}{3}}}$.

Find the radius of curvature of the following curves at the indicated points; draw each curve and construct the appropriate circle of curvature.

6. $y^2 = x^3$ at the point $(4, 8)$. *Ans.* $R = \frac{80\sqrt{10}}{3}$.

7. $x^2 = 4ay$ at the point $(0, 0)$. *Ans.* $R = 2a$.

8. $b^2x^2 - a^2y^2 = a^2b^2$ at the point (x_1, y_1). *Ans.* $R = \frac{(b^4x_1 + a^4y_1)^{3/2}}{a^4b^4}$.

9. $y = \ln x$ at the point $(1, 0)$. *Ans.* $R = 2\sqrt{2}$.

10. $y = \sin x$ at the point $\left(\frac{\pi}{2}, 1\right)$. *Ans.* $R = 1$.

11. $\left.\begin{array}{l} x = a\cos^3 t \\ y = a\sin^3 t \end{array}\right\}$ for $t = t_1$. *Ans.* $R = 3a \sin t \cos t$.

Find the radius of curvature of the indicated curves:

12. $\left.\begin{array}{l} x = 3t^2 \\ y = 3t - t^3 \end{array}\right\}$ for $t = 1$. *Ans.* $R = 6$.

13. Circle $\varrho = a \sin\theta$. *Ans.* $R = \frac{a}{2}$.

14. Spiral of Archimedes $\varrho = a\theta$. *Ans.* $R = \frac{(\varrho^2+a^2)}{\varrho^2+2a^2}$.

15. Cardioid $\varrho = a(1-\cos\theta)$. *Ans.* $R = \frac{2}{3}\sqrt{2a\varrho}$.

16. Lemniscate $\varrho^2 = a^2\cos 2\theta$. *Ans.* $R = \frac{a^2}{3\varrho}$.

17. Parabola $\varrho = a\sec^2\frac{\theta}{2}$. *Ans.* $R = 2a\sec^3\frac{\theta}{2}$.

18. $\varrho = a\sin^3\frac{\theta}{3}$. *Ans.* $R = \frac{3}{4}a\sin^2\frac{\theta}{3}$.

Find the points of curves at which the radius of curvature is a minimum:

19. $y = \ln x$. *Ans.* $\left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\ln 2\right)$.

20. $y = e^x$. *Ans.* $\left(-\frac{1}{2}\ln 2, \frac{\sqrt{2}}{2}\right)$.

21. $\sqrt{x}+\sqrt{y}=\sqrt{a}$. *Ans.* $\left(\frac{a}{4}, \frac{a}{4}\right)$.

22. $y = a\ln\left(1-\frac{x^2}{a^2}\right)$. *Ans.* At the point $(0, 0)$ $R = \frac{a}{2}$.

(d) $r^2 = a^2\cos 2\theta$. Find the equation of the pedal curve and the radius of curvature.

(e) Find the radius of curvature of $x = \log t, y = (t+t^{-1})/2$

(f) Show that the parabolas $y = 1+x-x^2$ and $x = 1+y-y^2$ have same circle of curvature at $(1,1)$

(g) Three more

42. Show that the radius of a curvature of a cycloid at any one of its points is twice the length of the normal at that point.

43. Write the equation of the circle of curvature of the parabola $y = x^2$ at the point (1, 1). *Ans.* $(x+4)^2+\left(y-\frac{7}{2}\right)^2=\frac{125}{4}$.

44. Write the equation of the circle of curvature of the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$. *Ans.* $\left(x-\frac{\pi-10}{4}\right)^2+\left(y-\frac{9}{4}\right)^2=\frac{125}{16}$.

(h) The radius of curvature of $x^{2/3}+y^{2/3}=a^{2/3}$ is three times the length of the pedal from origin.