Three proofs of Erdős-Szekeres lemma

Erdős-Szekeres lemma: A sequence of $n^2 + 1$ real numbers contains a monotonic subsequence of length $n + 1$.

First Proof. (direct argument)

Let the sequence be $\langle x_1, x_2, \ldots, x_{n^2+1} \rangle$. We associate an ordered pair (a_i, b_i) for each $1 \leq i \leq n^2 + 1$ such that a_i is the length of the longest monotonically increasing sequence ending at x_i and b_i is the length of the longest monotonically decreasing sequence ending at x_i. We might note that if $i < j$ then $(a_i, b_i) \neq (a_j, b_j)$. If $x_i < x_j$, then a_j is at least one more than a_i. Else if, $x_i > x_j$, then b_j is at least one more than b_i. By pigeon hole principle, there is at least one ordered pair other than $\{(a, b) \,|\, 1 \leq a \leq n, 1 \leq b \leq n\}$.

Second Proof. (based structure theorem of finite posets)

Let $X = \{1, 2, 3, \ldots, n^2 + 1\}$. Lets define an order $i \preceq j$ if $i \leq j$ and $x_i \leq x_j$ where $1 \leq i, j \leq n^2 + 1$ to get the poset $(X, \preceq)$. A chain $i_1 \preceq i_2 \preceq \cdots \preceq i_m$ in $(X, \preceq)$ corresponds to a monotonically increasing subsequence $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_m}$ (as $i_1 < i_2 < \cdots < i_m$). Similarly, an antichain in $(X, \preceq)$ corresponds to a monotonically decreasing sequence. By structure theorem, $\alpha(X, \preceq) \cdot \omega(X, \preceq) \geq n^2 + 1$ where $\alpha(X, \preceq)$ is the length of the longest chain and $\omega(X, \preceq)$ is the length of the longest antichain. Hence, $\alpha(X, \preceq) > n$ or $\omega(X, \preceq) > n$.

Third Proof. (based on Dilworth's theorem)

Suppose define the order as in the second proof. If the length of every chain does not exceed n, then, there must be more than n chains. By dilworth's theorem, we have an antichain of length greater than n.