Parity arguments in problem solving

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- Parity and apply it in unexpected situations

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- Methods of proof

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- Realize that mathematics is an iterative process and beautiful. It is not a system of theorems, formulae or results alone.

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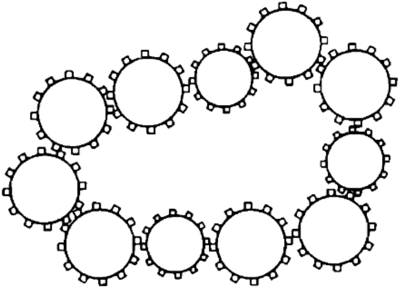
- Does parity change when we add an even number to number?

Problem.

Eleven gears are arranged on a plane. Can they rotate simultaneously?



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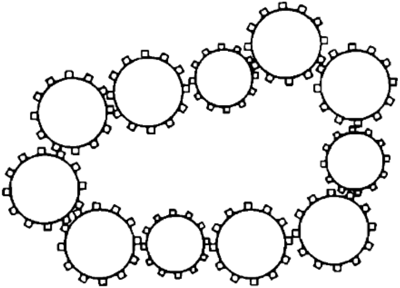

Answer. Any two neighbouring gears turn in opposite directions. So, 11 gears cannot rotate simultaneously. A system of odd number of gears cannot rotate simultaneously.

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Answer. Product of $2k$ integers is 1. Then, their sum is never zero if k is odd.

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Answer.

$$\text{odd} \pm \text{odd} = \text{even}$$
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$$\text{even} \pm \text{even} = \text{even}$$

There are five even and five odd numbers. Adding an even number does not change the parity of a sum. Five odd numbers add up to an odd number. So, the sum can never be zero.

Problem.

A grasshopper jumps along a line. His first jump takes him 1 cm, his second 2 cm, and so on. Each jump can take him to the right or to the left. Show that after 2013 jumps the grasshopper cannot return to the point at which he started.

Answer. This equivalent to whether some sum $\pm 1 \pm 2 \pm \cdots \pm 2013 = 0$. Thinking on the same lines as the previous problem, it is not possible.

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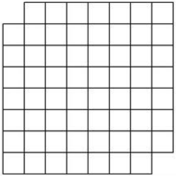
Answer. No, 2 does not divide 15.

Problem.

Can we cover a chessboard (8×8) with lower-right square and top-left square removed with dominoes?



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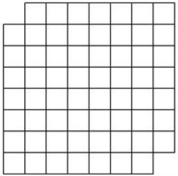
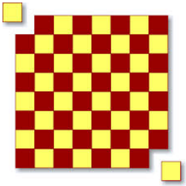


Answer. Every domino occupies a black and a white square. We have removed both black or white squares. So we should not be able to cover with dominoes.



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Nine points are chosen along line AB, all lying outside of segment AB. Prove that the sum of the distances from these points to point A is not equal to the sum of the distances of these points to point B.

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Answer. Let $dist(P, Q)$ denote the distance between point P and point Q. For any of the nine points, say P, $dist(P, A) - dist(P, B) = \pm \mid AB \mid$ depending on whether the point P is closer to B or A respectively (where $\mid AB \mid$ denotes the length of the segment AB). Since, there are odd(nine) number of points, the sum of all differences(from all nine points) cannot be zero.

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A snail crawls along a plane with constant velocity, turning through a right angle every 15 minutes. Show that the snail can return to its starting point only after a whole number of hours.

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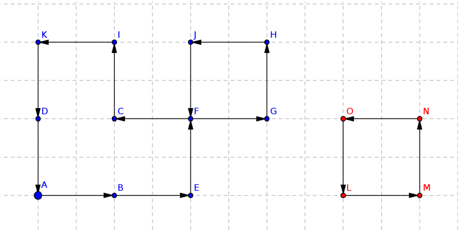
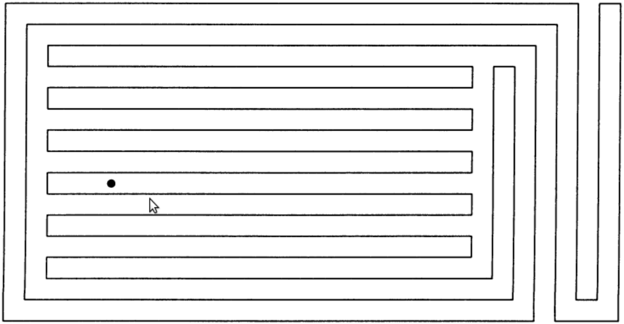
Answer. A snail has to retrace the direction to come back to the starting point. But such a retrace is possible if it turns a right angle twice or even number of times. So it takes 60 minutes to make a simple loop. Any other complicated path can be treated as an equivalent combination of `simple loops'.



Problem.

Is the point inside the figure or outside? What's your idea to figure it out?



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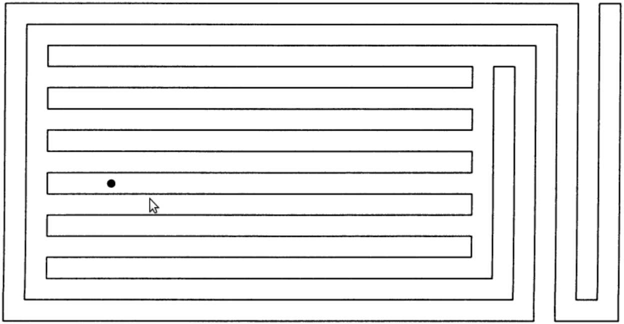

Answer. Count the number of times the a ray starting from the point crosses the boundary. This is a crude form of Jordan curve theorem.

Problem.

Suppose that $x_1, x_2, \ldots x_n$ is a rearrangement of the numbers $1, 2, \ldots, n$ where n is odd. Show that the product $(x_1 - 1)(x_2 - 2) \cdots (x_n - n)$ is even.

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Answer. Since n is odd, there will be $\frac{n+1}{2}$ odd numbers and $\frac{n-1}{2}$ even numbers. That is, there is one less even number compared to the number of odd numbers. Hence, one of the odd numbers ends up being in an odd position after the rearrangement. This makes the product even.

Problem.

The sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not an integer for all $n > 1$

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Answer. Suppose

$$m=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

be an integer for some n. Let k be the highest power of 2 such that $2^k \leq n$. Then, the lcm of $\{1,2,\ldots,n\}$ is $\ell = 2^k c$, for some odd number c. Multiplying the previous equation throughout by ℓ gives, sum of n terms among which exactly one is odd (on LHS) and $m\ell$ on the RHS. This makes LHS odd and RHS even. A contradiction!

More challenging problems

- In a tournament with 7 teams. Is it possible to schedule games so that each team plays exactly 5 games ?
- Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.
- Of 101 coins, 50 are counterfeit, and they differ from the genuine coins in weight by 1 gram. We have a balance which shows the difference in weight between the objects placed on each pan. You are allowed to choose one coin randomly, and you are asked to find out whether it is counterfeit. Can you do this in one weighing?
- In a 6×6 board, all but one corner blue square are painted white. You are allowed to repaint any column or any row in the chart (i.e., you can select any row or column and flip the colour of all squares within that line). Is it possible to attain an entirely white chart by using only the permitted operations?

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