Lecture 9: Introduction to functions - III

odd and even functions, finite maps, composition of two functions

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Even and Odd functions

A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

Exercise.

Which of the functions are even/odd ?

a.

$$x \sin^2 x - \tan^3 x$$

b.

$$\frac{e^x + 1}{e^x - 1}$$

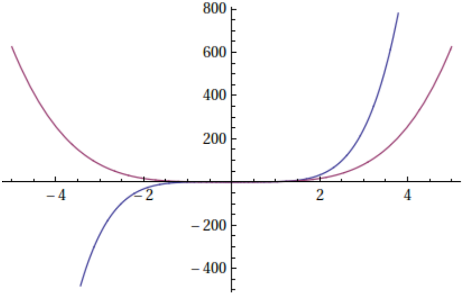
c.

$$\sin x + \cos x$$

Answer. a. odd, b. odd, c. neither odd nor even

Odd and Even functions, a graphical perspective

Even functions are symmetric about y-axis. Odd functions are cross-symmetric about x-axis. Consider x^4 and x^5.



Challenging Problem

Prove that any function $f : \mathbb{R} \to \mathbb{R}$ can be written as sum of two functions such that one of them is even and the other is odd.

Functions from finite sets to finite sets

Let $n(A)$ denote the number of elements of a finite set A. Let A and B be finite sets.

Exercise. If $f : A \to B$ is injective. What can we say about $n(A)$ and $n(B)$?

Answer. $n(A) \leq n(B)$

Exercise. If $f : A \to B$ is surjective. What can we say about $n(A)$ and $n(B)$?

Answer. $n(A) \geq n(B)$

Exercise. If $f : A \to B$ is bijective, then ...

Answer. $n(A) = n(B)$

Corollary: If $f : A \to A$, then f is injective $\Leftrightarrow$ f is surjective.

Problem.

Let A be set with odd number of elements such that $f : A \to A$, $f(f(x)) = x \quad \forall x \in A$. Then,

(a) f has to be bijective

(b) f is a constant function

(c) $\exists\, y \in A$ such that $f(y) = y$ (some element maps to itself)

(d) If A had even number of elements, then the answer for the problem would not differ.

Solution.

- Note that if $x \to y$, then $y \to x$. Since, $n(A)$ is odd, $\exists\, y \in A$ such that $f(y) = y$.
- Further, define g such that g takes y to x whenever f takes x to y. Hence, $g = f^{-1}$. This means that f has to be bijective. Aliter: Prove that the f has to be either injective or surjective.
- Let $f : \{a, b, c\} \to \{a, b, c\}$ such that $f(a) = b, f(b) = c, f(c) = c$. This satisfies all conditions and is not a constant function.
- If A had even number of elements, (c) would fail. So the answer differs.

Answer. (a),(c)

Composition of functions

The function $f(x) = \sqrt{x+1}$ can be viewed as a combination of two functions. Consider $h(\#) = \# + 1$ and $g(\#) = \sqrt{\#}$. We can write $f(x) = goh(x) = g\,(h(x))$.

On the other hand, $hog(x) = h(g(x)) = \sqrt{x} + 1$.

<u>Caveat</u>: We need to be careful about the domain and co-domain of the composed function.Note that $h : \mathbb{R} \to \mathbb{R}$ and $g : [0, \infty) \to [0, \infty)$. But,
$f : [-1, \infty) \to [0, \infty)$

Exercise.

$f = \sqrt{x}, g = \mid x \mid$, discuss fog and gof, their domain and range.

Solution. $fog(x) = \sqrt{\mid x \mid}$. Hence, $D(fog(x)) = \mathbb{R}$ and $Im(fog(x)) = [0, \infty)$.
$gof(x) = \mid \sqrt{x} \mid$. Hence, $D(gof(x)) = Im(gof(x)) = [0, \infty)$

Problems based on Composition of functions

Problem.

Let $f\,o\,f(x) = 0 \quad \forall\, x \in D(f)$. Then,
(a) $\exists\, y \in D(f)$ such that $f(y) = 0$
(b) $f(0) = 0$
(c) $f\,o\,f\,o\,f(x) = 0 \quad \forall\, x \in D(f)$
(d) None of these

Answer. a,b,c Solution. (a) is true. Lets assume $0 \rightarrow a$ where $a \neq 0$. Then, $0 \rightarrow a \rightarrow 0$ as $f(f(0)) = 0$. Note that $a \rightarrow 0 \rightarrow 0$ as $f(f(a)) = 0$. The blue part in the previous equation contradicts the assumption. Hence, $0 \rightarrow 0$. (c) is equivalent to (b).

Problem.

$f, g : \mathbb{R} \rightarrow \mathbb{R}$, where $f(x) = x + 1$ and $g(x) = x^2 - 1$. For which x does $gof(x) = 10$

Solution. $gof(x) = (x+1)^2 - 1 = 10$. Solving the quadratic yields,
$x = \pm\sqrt{11} - 1$

More Problems based on Composition of functions

Problem.

$fog(x) = \left(\sin\sqrt{x}\right)^2, gof(x) = \mid \sin x \mid$. Find a possibility of f and g.

Solution. Note that $\sqrt{2^2} = \sqrt{(-2)^2} = 2$. In general, $\sqrt{a^2} = \mid a \mid$. Some thought must convince us that $f(\#) = (\sin \#)^2$ and $g(\#) = \sqrt{\#}$ is a possibility.

Problem.

Let $f : A \to B$, $g : B \to C$. Prove that if f and g are injective(surjective), then fog is injective(surjective). When is fog bijective? (all of this is supposing fog exists and $fog : A \to C$)

Answer. fog is bijective when f is injective and g is surjective.

Challenging Problem

Find an bijective fog such that f is injective and g is surjective but neither f nor g are bijective.