Solutions to problems of week 8

1. Let A be a finite set and $f: A \to A$ be surjective. Prove that $\exists\, n \in \mathbb{N}$ such that
$$f_n(x) := \underbrace{f \circ f \cdots \circ f}_{n}(x) = x \quad \forall\, x \in A.$$

Solution: We shall first prove that for every $x \in A$, $\exists\, n \in \mathbb{N}$ such that $f_n(x) = x$

Since, f is surjection from ~~the~~ a finite set A to itself, f is a bijection.

Claim: $\{f(x), f_2(x), \ldots, f_n(x)\}$ are all distinct except when $f(x) = x$ (we mean, $f_{k_1}(x) \neq f_{k_2}(x)$ whenever $k_1 \neq k_2$)

~~Suppose if $f_r(x) = f_\ell(x)$ for some $n \geq r > \ell$, then $f_{r-1}(x) =$~~

If any two elements in the list are equal, then the map fails to be injective.

Hence, fore every $x \in A$, $\exists$ n (depending on x) such that $f_n(x) = x$.

Let $f_{n_1}(x_1) = x_1$

$f_{n_2}(x_2) = x_2$

$\vdots$

Let $l := \text{lcm}(n_1, n_2, \ldots)$

Then,

$$f_l(x) = x \quad \forall x \in A.$$

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2. Let A be a finite set. and $f : A \to A$ $\exists$ $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $f_n(x) = a$ $\forall x \in A$

Then, prove that $f(a) = a$

<u>Solution.</u>

$$f_n(a) = a$$

$$f_{n+1}(a) = a$$

$$\Rightarrow f(a) = a$$

3. Let A be a finite set and $f : A \to A$.

$\forall\, x \in A$, $\exists\, n \in \mathbb{N}$ (depending on x)

such that $f_n(x) = x$. Then f is bijective.

Solution. Since f is a map from finite set A to itself, its enough to prove that f is surjective. If f was not surjective, that $f_n(y) \neq y$ for any $n \in \mathbb{N}$ for some y.

Hence, f is surjective. Hence, f is bijective.

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4. $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ with $f((m,n)) = 6m - 9n$

Find the type of the function and its range (image).

Solution: Suppose,

$$f(m_1, n_1) = f(m_2, n_2)$$

$$\Rightarrow \; 3(2m_1 - 3n_1) = 3(2m_2 - 3n_2)$$

$$\Rightarrow \; 2(m_1 - m_2) = 3(n_1 - n_2)$$

choose $m_1 - m_2 = 3$ and $n_1 - n_2 = 2$

let $m_1 = 4$, $m_2 = 1$ and $n_1 = 3$, $n_2 = 1$

Then, $f(4,3) = 6 \times 4 - 9 \times 3 = -3$

$f(1,1) = 6 - 9 = -3$

Hence, f is not injective.

To prove or disprove its surjective:

$$f(m,n) = 3(2m-3n)$$

Always gives a multiple of 3. Hence, not surjective.

Range of the function $= \{3K, K \in \mathbb{Z}\}$

(all multiples of 3)

Choose $m = 2K$, $n = K$.

Then,

$$f(m,n) = 3(4K - 3K) = 3K$$

Therefore, we get all multiples of 3.

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4 Prove that $\mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (x-y, x+y)$ is bijective.

Solution To test injectivity:

Let $f(x_1, y_1) = f(x_2, y_2)$

Then, $(x_1 - y_1, x_1 + y_1) = (x_2 - y_2, x_2 + y_2)$

$\Rightarrow$ $x_1 - y_1 = x_2 - y_2$ and

$x_1 + y_1 = x_2 + y_2$

Adding the two equations.

$$2x_1 = 2x_2$$

$$\Rightarrow \quad x_1 = x_2$$

On substracting them,

$$y_1 = y_2$$

Hence, $(x_1, y_1) = (x_2, y_2)$

$\Rightarrow$ f is injective.

Note that if $(l, m) \rightarrow (x, y)$

Then $l - m = x$ and
$l + m = y$

$$\Rightarrow \quad 2l = x + y \quad \Rightarrow \quad l = \frac{x+y}{2}$$

Similarly, $m = \frac{x-y}{2}$

Hence,

$$\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \rightarrow (x, y).$$

6. $$f(x) = x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}$$

Solution: Note that $f(x) = x + \dfrac{1}{f(x)}$

$$\Rightarrow f^2(x) - x f(x) - 1 = 0$$

$$\Rightarrow f(x) = \frac{x \pm \sqrt{x^2+4}}{2}$$