Lecture 8: Introduction to functions - II

inverse of a function, more examples, plotting a function, simple transformations

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http://bit.ly/trig2013

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Solution. Note that $1 \leftrightarrow 2, 3 \leftrightarrow 4, 5 \leftrightarrow 6, \ldots$. Hence, f is a bijection and inverse of f is f itself.

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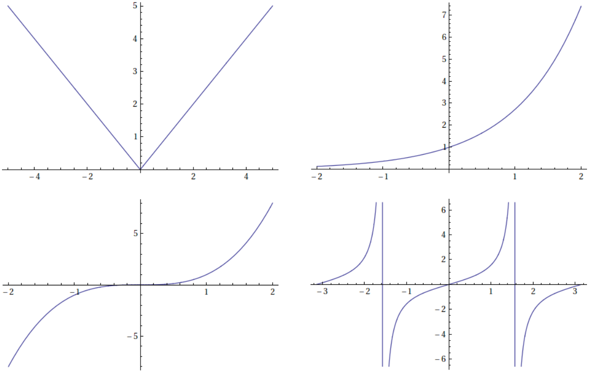

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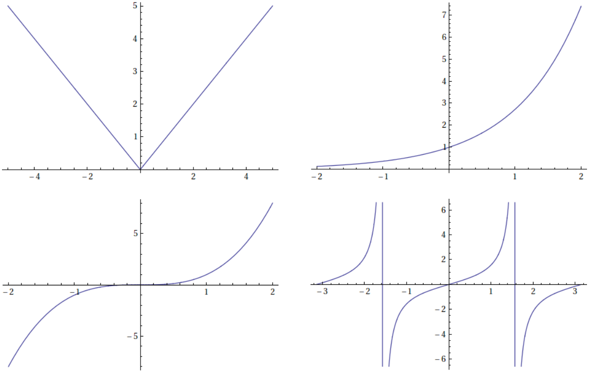

$a \to 4, b \to 3, c \to 2, d \to 1$

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- Horizontal line test: If a horizontal line touches two different points of the plot, it must not be injective.

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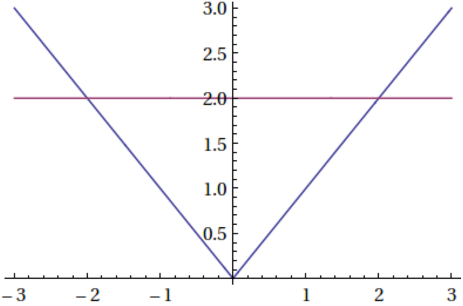

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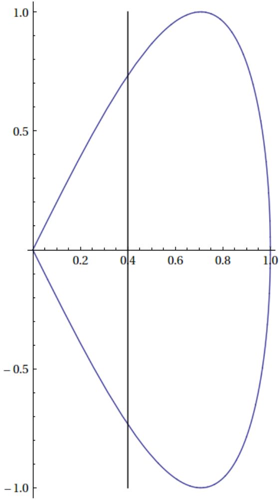

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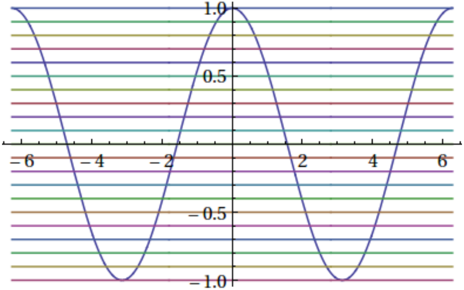
- Exercise. How would we test for a surjective function graphically? Every horizontal line should intersect or touch the graph of the function.

  Consider, $f : [-2\pi, 2\pi] \to [-1, 1]\,,\ f(x) = \cos x$



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Hence the domain is $\{x, x > 3\} \cup \{x, x < 1\}$
In interval notation, we write $(-\infty, 1) \cup (3, \infty)$

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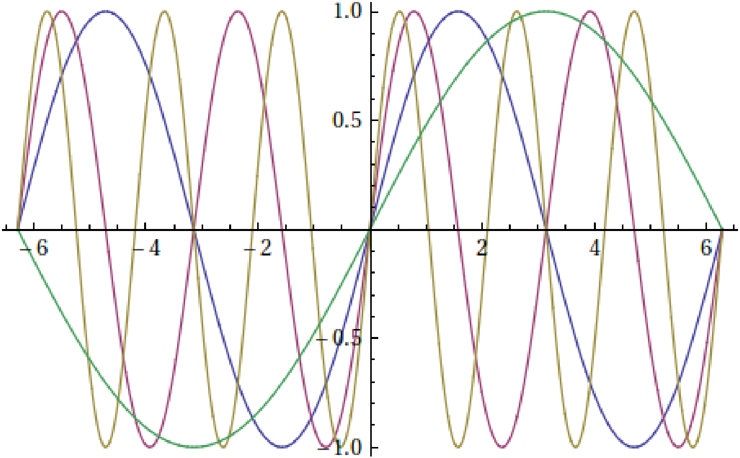

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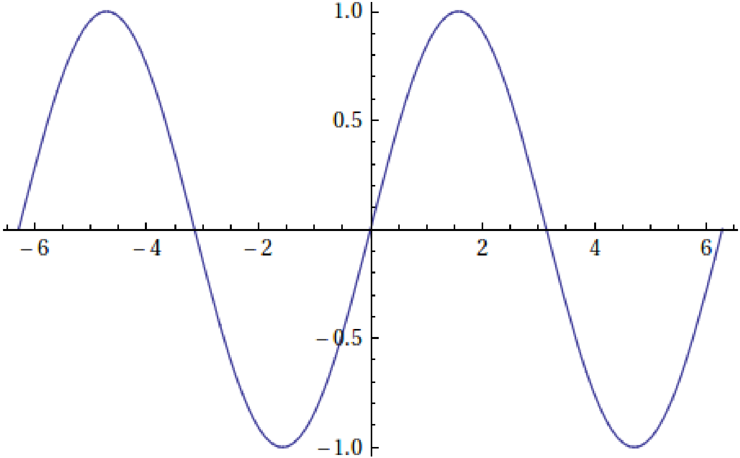

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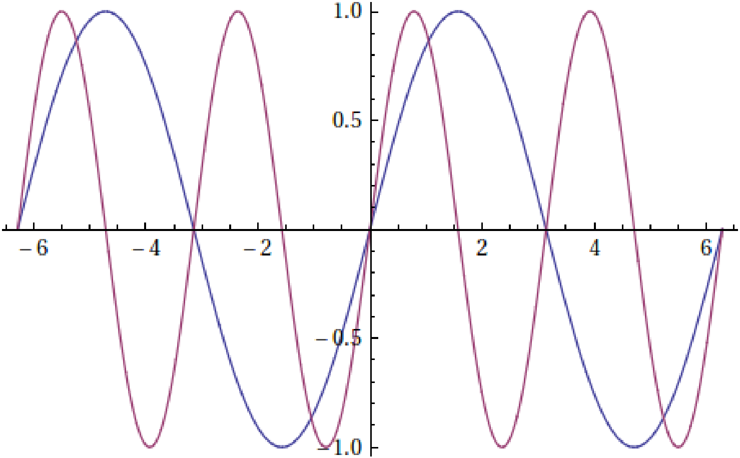

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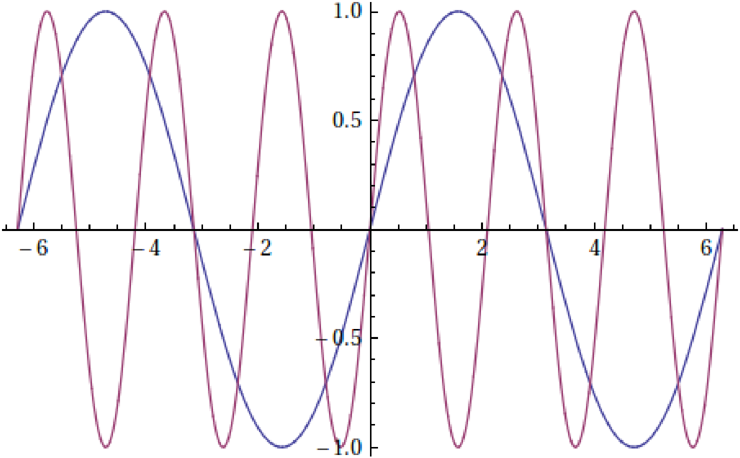

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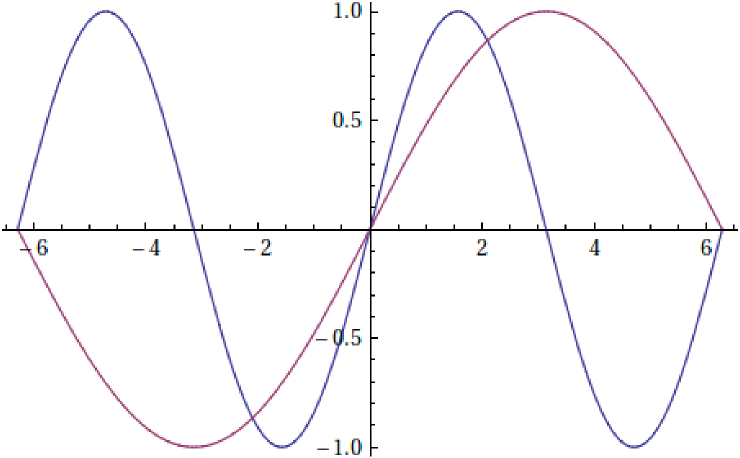

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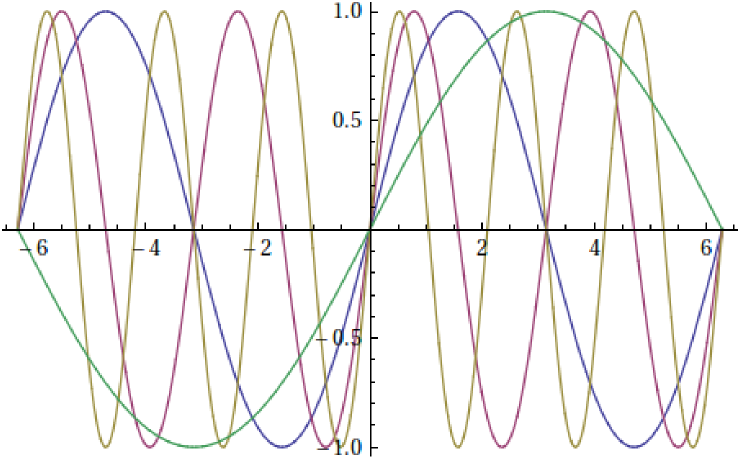

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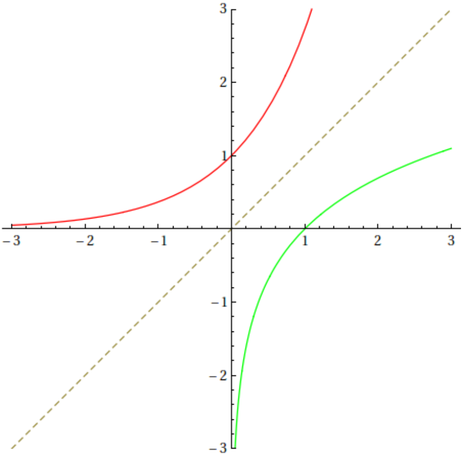

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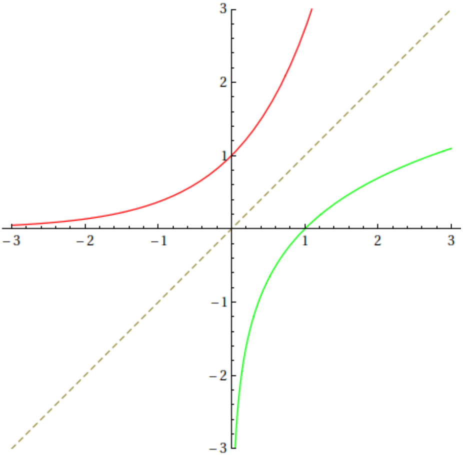

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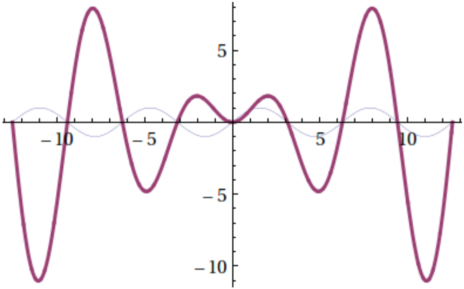

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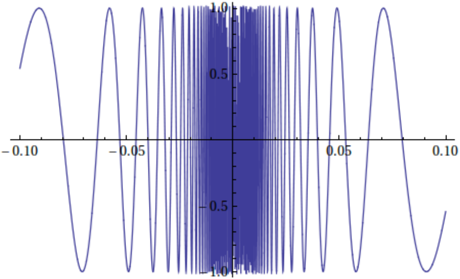

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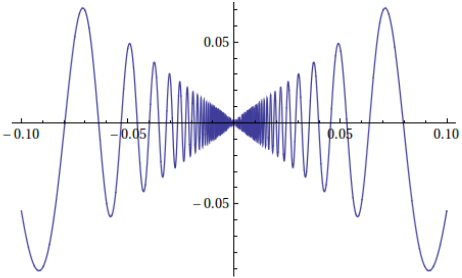

Plot $x^2 \sin 1/x$

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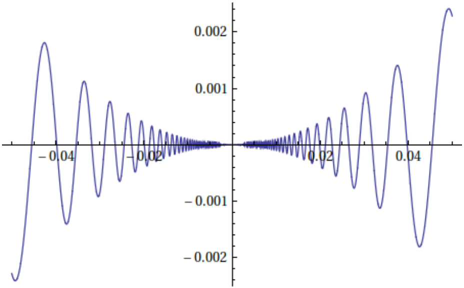
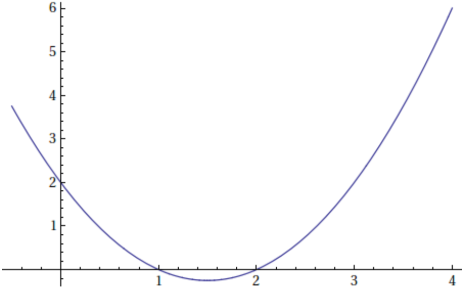


Transforming a function

Lets study how a plot transforms when the function altered. Here is the plot of $f(x) = x^2 - 3x + 2$



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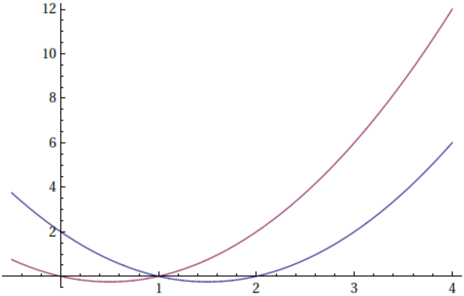

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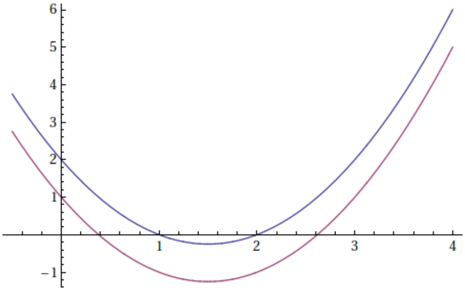

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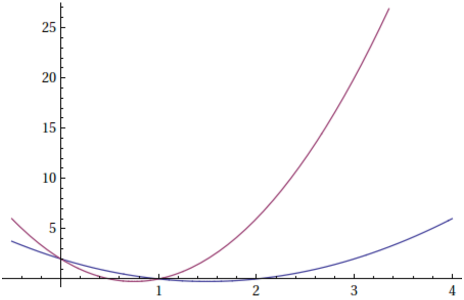

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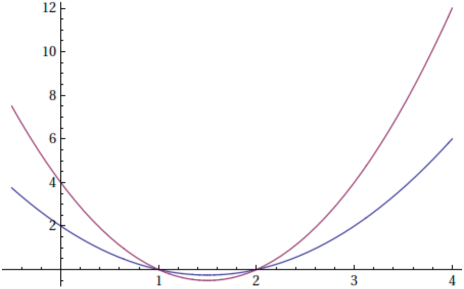

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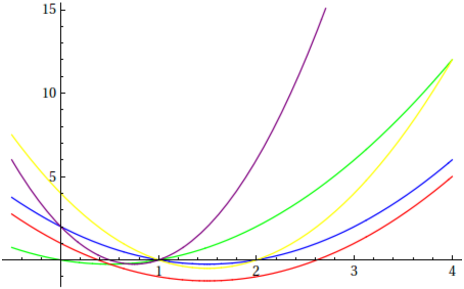
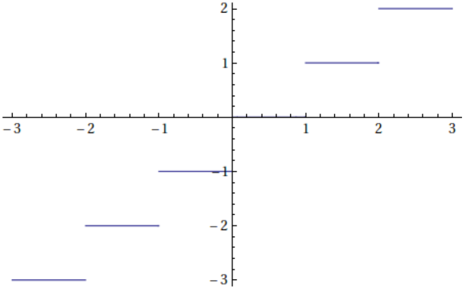
$f(x) - 1 = x^2 - 3x + 1$, $f(2x) = (2x)^2 - 3(2x) + 2$, $2f(x) = 2(x^2 - 3x + 2)$



Some special functions

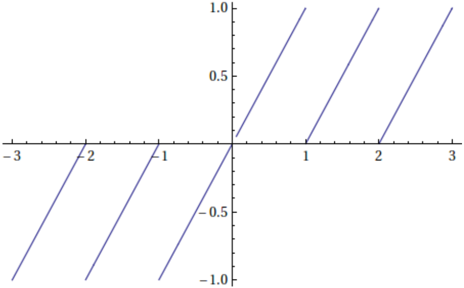
Greatest integer function or the floor function

$f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = [x] = \lfloor x \rfloor$=greatest integer less then or equal to x

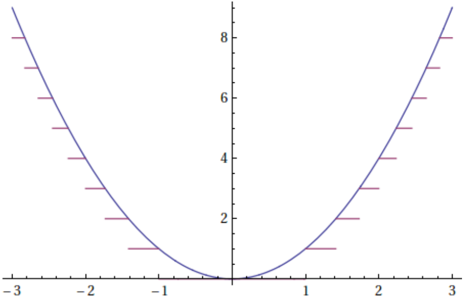


Fractional part function $f : \mathbb{R} \to \mathbb{R}$ such that

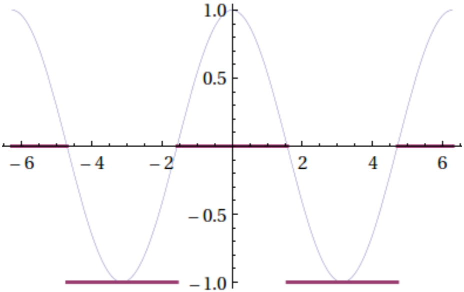
$$f(x) = \begin{cases} x - \lfloor x \rfloor & \text{if } x \geq 0, \\ \lfloor x \rfloor - x & \text{if } x < 0 \end{cases}$$



Plot of $\lfloor x^2 \rfloor$



Plot of $\lfloor \cos x \rfloor$



Problems

Let $fof(x)$ denote $f(f(x))$.

- Let A be a finite set and $f : A \to A$ be surjective. Prove that $\exists\, n \in \mathbb{N}$ such that $f_n(x) := \underbrace{fofo\ldots of}_{n}(x) = x \quad \forall\, x \in A$
- Let A be a finite set and $f : A \to A$. $\exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geq n_0 \quad f_n(x) := \underbrace{fofo\ldots of}_{n}(x) = a \quad \forall\, x \in A$. Prove that $f(a) = a$
- Let A be a finite set and $f : A \to A$. If for every $x \in A$, there exists a $n \in \mathbb{N}$ (depending on x) such that $f_n(x) = x$, prove that f is bijective.
- $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined as $f(m, n) = 6m - 9n$. Decide the type of the function and its range.
- Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (x - y, x + y)$ is bijective. Obtain its inverse.
-
$$f(x) := x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{\ddots}}}$$
Supposing that RHS makes sense, find f, its domain and range.