Lecture 7: Introduction to functions - I

basic definitions, examples, types of functions

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'Functions' is an invitation to higher mathematics. Topics in calculus depend on sound understanding of functions.

Number systems

- The set $\{1, 2, 3, \dots\}$ is called the set of <u>natural</u> numbers (denoted by symbol $\mathbb{N}$). There is nothing like the biggest number in this set. But the subtraction of two natural numbers need not be natural.
- The set $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is called the set of <u>integers</u> $(\mathbb{Z})$. There is nothing like the biggest number or the smallest in this set. But the division of two integers need not be an integer.
- The set of all fractions, that is $\{\frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\}$ is called the set of <u>rational</u> numbers (fractions) $(\mathbb{Q})$. The rational numbers are <u>closely packed</u>. There is nothing like the next rational number. When written as a decimal, a rational number is either a terminating decimal or recurring.
- A number is said to be <u>irrational</u> if it's decimal representation is non-terminating and non-recurring. The set of all rational and irrational numbers form the real number set $(\mathbb{R})$.
- Complex numbers ... we shall discuss them sometime later.

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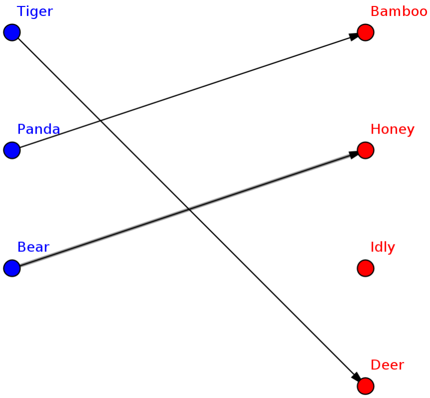


- Here, Domain of this function is {Tiger, Panda, Bear}, the co-domain is {Bamboo, Honey, Idly, Deer}, $Im(f) =$ {Bamboo, Honey, Deer}
- A function consists of three parts: the domain, the co-domain, the information(arrows) in the function itself(which decides the image or range, $Im(f)$)

Conditions for a set to be a function

The function on the previous slide may be written as a set as {domain, co-domain, arrows}=

{ {Tiger, Panda, Bear},{Bamboo, Honey, Idly, Deer},{{Tiger,Deer},{Panda,Bamboo},{Bear,honey}}}

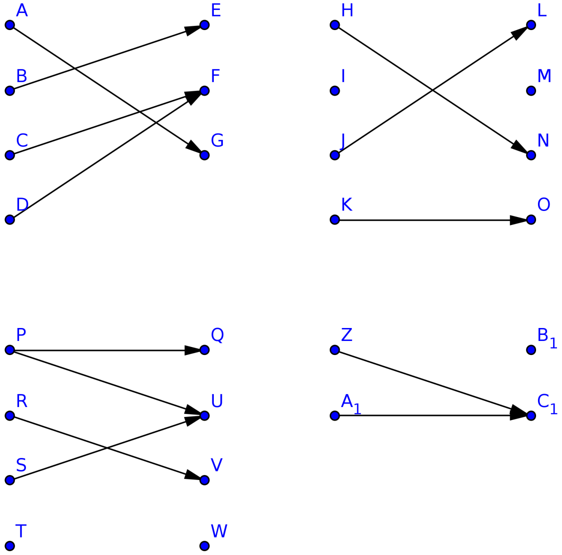
In function notation,
f : domain $\rightarrow$ co-domain with f(Tiger)=Deer, f(Panda)=Bamboo, f(Bear)=Honey

Conditions:

1. Every element in the domain is mapped to some element of the co-domain

2. Every element in the domain maps to exactly one element of the co-domain

Which of these are functions:



Examples

1. Let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$, defined as f(n)=Number of prime numbers less than or equal to n. A few function values are:

$$f(1) = 0$$
$$f(2) = 1$$
$$f(3) = 2$$
$$f(4) = 2$$
$$f(5) = 3$$
$$\cdots$$

.

2. Let $f : \mathbb{N} \to \mathbb{Z}$, such that

$$f(x) = \begin{cases} 1 & \text{if x is odd,} \\ -1 & \text{if x is even} \end{cases}$$

Find $Im(f)$

3. Let

$$f(x) = \frac{x+1}{x-1}$$

a. Can $\mathbb{R}$ be the domian of f?

If $x = 1$, the function is undefined. Hence, the domain of f ie., $D(f) = \mathbb{R} \setminus \{1\}$

b. Can $\mathbb{R}$ be the image(range) of f?

On simplification, $\frac{f(x)+1}{f(x)-1} = x$. This shows that f could have never been 1.

4. We define,

$$f(x) = \mid x \mid = \begin{cases} x & \text{if } x > 0, \\ -x & \text{if } x \leq 0 \end{cases}$$

what can we say about the domian and image of $f(x) = \sqrt{1- \mid 1-x \mid}$

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\begin{aligned}
1- \mid 1-x \mid &\geq 0 \\
\mid 1-x \mid &\leq 1 \\
\Rightarrow -1 \leq 1-x &\leq 1 \\
\Rightarrow -2 \leq -x &\leq 0 \\
\Rightarrow 2 \geq x &\geq 0
\end{aligned}
$$

Since, $\mid 1-x \mid$ is positive, $1- \mid 1-x \mid$ is less than 1(but greater than 0). Hence, $\sqrt{1- \mid 1-x \mid}$ lies between(and including) 0 and 1.

Problem. Let $f : \mathbb{Z} \to \mathbb{R}$ such that $f(x) = f(x+5) = f(x+13) \, \forall \, x \in \mathbb{Z}$. Prove that $f(x) = k \, \forall \, x \in \mathbb{R}$(a constant function)

Solution. Note that $f(x) = f(x+5) = f(x+10) = \cdots = f(x+25)$ and $f(x) = f(x+13) = f(x+26)$. This means any two neighboring function values are same. Hence, f is a constant function.

Problem. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(2x) = f(x), \forall x \in \mathbb{R}$ and $f(x) = 1, \forall \frac{-1}{10^{10}} < x < \frac{1}{10^{10}}$. What can we tell about f?

Solution. Lets choose x such that it does not lie between $\frac{-1}{10^{10}}$ and $\frac{1}{10^{10}}$. Then, eventually for some $n \in \mathbb{N}$, $\frac{x}{2^n}$ lies between $\frac{-1}{10^{10}}$ and $\frac{1}{10^{10}}$.

Problem. Let $f : \mathbb{Z} \to \mathbb{R}$ such that $f(x) = \frac{f(x-1)+f(x+1)}{2}$ and $f(0) = 0$. Then,

(a) all function values are positive except at $x = 0$

(b) if $f(10) = 1$, then $f(-5)$ is negative

(c) the function values form an increasing arithmetic progression

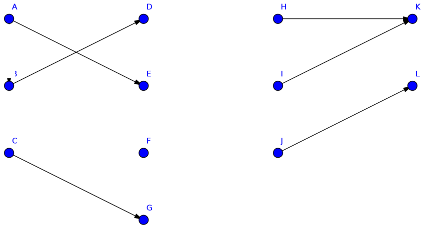
(d) none of these

(a) fails if $f(x) \equiv 0$. If $f(1) > 0$, note that $f(-1) < 0$ and vice versa. So, all positive integers get mapped to positive(negative) reals and vice versa. Hence, (b) is correct. (c) is not true, as we may have a decreasing sequence too.

Injective or one-one or into function

A function is said to be injective if no two elements of the domian map to the same element of the co-domain. In other words,
f is injective $\Leftrightarrow (f(a) = f(b) \Rightarrow a = b)$

Exercise. Are the functions injective?



Exercise. Is $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \frac{1+x}{x}$ injective?

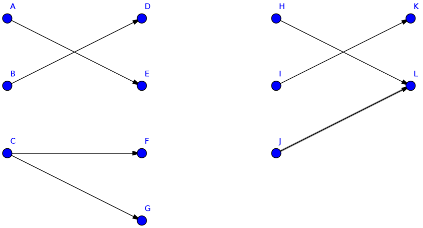
Solution. Yes,

$$\frac{1 + x_1}{x_1} = \frac{1 + x_2}{x_2} \Rightarrow x_1 = x_2$$

Surjective or onto function

A function is said to be surjective if every element of the co-domain has a pre image (is mapped by some element of the domain) In other words,
f is surjective $\Leftrightarrow \forall\, a \in$ Co-domian $\exists\, b \in$ Domain such that $f(b) = a$

Exercise. Are the functions surjective?



Exercise. Is $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \frac{1+x}{x}$ surjective?

Solution. We find that $x = \frac{1}{f(x)-1}$. Hence, $f(x) \neq 1\, \forall\, x \in \mathbb{R}$. Hence, f is not surjective