Lecture 6: Transformation formulae (product $\Longleftrightarrow$ sum)

Srikanth K S

http://bit.ly/trig2013

We would like to express $\sin a \cos b$ as a sum of two expressions.

We have,

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

and

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

Adding the two equations, we get

$$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$$

Similarly, subtracting the second equation from the first gives

$$\sin(a+b) - \sin(a-b) = 2 \cos a \sin b$$

Exercise. Express $\cos a \cos b$ and $\sin a \sin b$ as sum of two expressions.

Solution. Consider

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

and

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Adding the two equations, we get

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$$

Subtracting the first equation from the second, we get

$$\cos(a-b) - \cos(a+b) = 2\sin a \sin b$$

$$\sin(a+b)+\sin(a-b)=2\sin a\cos b \tag{1}$$
$$\sin(a+b)-\sin(a-b)=2\cos a\sin b \tag{2}$$
$$\cos(a+b)+\cos(a-b)=2\cos a\cos b \tag{3}$$
$$\cos(a-b)-\cos(a+b)=2\sin a\sin b \tag{4}$$

Let $c := a+b$ and $d := a-b$. Then, solving for a and b gives $a = \frac{c+d}{2}$ and $b = \frac{c-d}{2}$
Rewriting all the equations on terms of c and d gives

$$\sin c+\sin d=2\sin\frac{c+d}{2}\cos\frac{c-d}{2} \tag{5}$$
$$\sin c-\sin d=2\cos\frac{c+d}{2}\sin\frac{c-d}{2} \tag{6}$$
$$\cos c+\cos d=2\cos\frac{c+d}{2}\cos\frac{c-d}{2} \tag{7}$$
$$\cos d-\cos c=2\sin\frac{c+d}{2}\sin\frac{c-d}{2} \tag{8}$$

Problem. Prove

$$\frac{\sin a + 2\sin 5a + \sin 9a}{\cos a + 2\cos 5a + \cos 9a} = \tan 5a$$

Solution.

$$\begin{aligned} LHS &= \frac{\sin a + \sin 9a + 2\sin 5a}{\cos a + \cos 9a + 2\cos 5a} \\ &= \frac{2\sin 5a\cos 4a + 2\sin 5a}{2\cos 5a\cos 4a + 2\cos 5a} \\ &= \frac{2\sin 5a(\cos 4a + 1)}{2\cos 5a(\cos 4a + 1)} \\ &= \tan 5a \qquad\qquad \text{assuming } \cos 4a + 1 \neq 0 \end{aligned}$$

Problem. $\sin 20 \sin 40 \sin 60 \sin 80 = 3/16$

Solution.

$$\begin{aligned}
\sin 20 \sin 40 \sin 60 \sin 80 &= \frac{\sqrt{3}}{2} \cos 70 \cos 50 \cos 10 \\
&= \frac{\sqrt{3}}{2} \left(\frac{\cos 20 + \cos 120}{2} \right) \cos 10 \\
&= \frac{\sqrt{3}}{2} \left(\frac{\cos 20 - \frac{1}{2}}{2} \right) \cos 10 \\
&= \frac{\sqrt{3}}{4} \left(2\cos^2 10 - 1 - \frac{1}{2} \right) \cos 10 \\
&= \frac{\sqrt{3}}{8} (4\cos^2 10 - 3) \cos 10 \\
&= \frac{\sqrt{3}}{8} (4\cos^3 10 - 3\cos 10) \\
&= \frac{\sqrt{3}}{8} \cos 30 \\
&= \frac{3}{16}
\end{aligned}$$

Problem. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\ldots\cos\frac{7\pi}{15}$

Solution. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$

$$= \frac{\sin\frac{8\pi}{15}}{8\sin\frac{\pi}{15}}\cos\frac{7\pi}{15}\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}$$
$$= \frac{\sin\frac{15\pi}{15}+\sin\frac{\pi}{15}}{16\sin\frac{\pi}{15}}\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}$$
$$= \frac{1}{16}\cos 36\cos 60\cos 72$$

Note that $\cos 36 = 1-2\sin^2 18 = 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}, \cos 60 = 1/2$ and $\cos 72 = \sin 18 = \frac{\sqrt{5}-1}{4}$

Hence, we get the answer as $\frac{1}{2^7}$

Problem. Prove that $\tan 6 \tan 42 \tan 66 \tan 78 = 1$

Solution. We shall prove that
$\sin 6 \sin 42 \sin 66 \sin 78 = \cos 6 \cos 42 \cos 66 \cos 78$

$$\frac{1}{4}(\cos 60 - \cos 72)(\cos 36 - \cos 120) = \frac{1}{4}(\cos 60 + \cos 72)(\cos 36 + \cos 120)$$

$$\frac{1}{4}\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right) = \frac{1}{4}\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)$$

Now prove the equality yourself ...

Problem. Let $a + b + c = 180$. Then, prove that $\cos 2a + \cos 2b + \cos 2c = -1 - 4\cos a \cos b \cos c$

Solution.

$$\begin{aligned}
\cos 2a + \cos 2b + \cos 2c &= 2\cos(a+b)\cos(a-b) + \cos 2c \\
&= 2\cos(180 - c)\cos(a-b) + 2\cos^2 c - 1 \\
&= -1 - 2\cos c\cos(a-b) + 2\cos^2 c \\
&= -1 - 2\cos c(\cos(a-b) - \cos c) \\
&= -1 - 2\cos c(\cos(a-b) + \cos(a+b)) \\
&= -1 - 4\cos c\cos a\cos b
\end{aligned}$$