Lecture 5: Multiple and Sub-Multiple angles

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Expand

$$\sin 2x$$

$$\begin{aligned}\sin 2x &= \sin(x+x)\\ &= \sin x \cos x + \cos x \sin x\\ &= 2\sin x \cos x\end{aligned}$$

$$\sin 4x$$

$$\begin{aligned}\sin 4x &= 2\sin 2x \cos 2x\\ &= 2(2\sin x \cos x)\cos 2x\end{aligned}$$

Similarly, expand

$$\sin x$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

Similarly, expand

$$\sin 100$$

$$\sin 100 = 2\sin 50 \cos 50$$

Expand

$$\cos 2x$$

$$\cos 2x = \cos(x + x)$$
$$= \cos x \cos x - \sin x \sin x$$
$$= \cos^2 x - \sin^2 x$$

writing in terms of cos alone,

$$= \cos^2 x - (1 - \cos^2 x)$$
$$= 2\cos^2 x - 1$$

writing in terms of sin alone,

$$= (1 - \sin^2 x) - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$

Expand in terms of cos alone: $\cos 4x$

$$\cos 4x = 2\cos^2(2x) - 1$$
$$= 2\left(2\cos^2 x - 1\right)^2 - 1$$

Rewrite the formulae for $\sin x$ and $\cos x$ in terms of $\cos 2x$:

$$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}$$

$$\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}$$

Use the above results to find $\cos 22.5$ and $\sin 7.5$

Expand

$$\tan 2x$$

$$\begin{aligned}\tan 2x &= \tan(x+x)\\ &= \frac{\tan x + \tan x}{1 - \tan x \tan x}\\ &= \frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

Using the above idea, expand $\tan 4x$ in terms of $\tan x$

$$\begin{aligned}\tan 4x &= \frac{2\tan 2x}{1-\tan^2 2x}\\ &= \frac{2\left(\frac{2\tan x}{1-\tan^2 x}\right)}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}\end{aligned}$$

Problems

Problem. Simplify

$$\sqrt{1 + \sin 2x}$$

Solution.

$$\begin{aligned}\sqrt{1 + \sin 2x} &= \sqrt{1 + 2\sin x \cos x} \\ &= \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} \\ &= \sin x + \cos x\end{aligned}$$

Similarly, we prove

$$\sqrt{1 - \sin 2x} = \pm(\sin x - \cos x)$$

Problem. Express $\sin 2x$ and $\cos 2x$ in terms of $\tan x$

Solution.

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ &= 2\sin x\cos x\frac{\cos x}{\cos x}\\ &= 2\frac{\sin x}{\cos x}cos^2x\\ &= 2\frac{\tan x}{\sec^2 x}\\ sin2x &= \frac{2\tan x}{1+\tan^2 x}\end{aligned}$$

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1\\ &= \frac{2}{\sec^2 x} - 1\\ &= \frac{2}{1+\tan^2 x} - 1\end{aligned}$$

Problem. Prove

$$1 + \tan x \tan\left(\frac{x}{2}\right) = \sec x$$

Solution. We shall write tan(x) in terms of its half angle

$$1 + \tan x \tan\left(\frac{x}{2}\right) = 1 + \frac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} \tan\frac{x}{2}$$

$$\begin{aligned}
1 + \tan x \tan\left(\frac{x}{2}\right) &= 1 + \frac{2\tan^2\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} \\
&= \frac{1 + \tan^2\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} \\
&= \frac{1}{\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}} \\
&= \frac{1}{\cos x} \\
&= \sec x
\end{aligned}$$

Problem. Expand $\sin 3x$ and $\cos 3x$ in terms of $\sin x$ and $\cos x$ respectively.

Solution.

$$\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= 2\sin x \cos x \cos x + (1 - 2\sin^2 x)\sin x \\
&= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x \\
&= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\
&= 3\sin x - 4\sin^3 x
\end{aligned}$$

Similarly, we can prove $\cos 3x = 4\cos^3 x - 3\cos x$

Problem.

$$4\sin^3 x\cos 3x + 4\cos^3 x\sin 3x = m\sin kx$$

Find the values of m and k.

Solution.

$$\begin{aligned} LHS &= 4\sin^3 x(4\cos^3 x - 3\cos x) + 4\cos^3 x(3\sin x - 4\sin^3 x) \\ &= 12\cos x\sin x(\cos^2 x - \sin^2 x) \\ &= 6(2\cos x\sin x)(\cos^2 x - \sin^2 x) \\ &= 6\sin 2x\cos 2x \\ &= 3(2\sin 2x\cos 2x) \\ &= 3\sin 4x \end{aligned}$$

Hence, $m = 3$ and $k = 4$.

Problem. Prove that

$$\cos 20 \cos 40 \cos 80 = 1/8$$

where all angles are in degrees.

Solution.

$$\begin{aligned}
\cos 20 \cos 40 \cos 80 &= \frac{1}{\sin 20} \sin 20 \cos 20 \cos 40 \cos 80 \\
&= \frac{1}{\sin 20} \frac{\sin 40}{2} \cos 40 \cos 80 \\
&= \frac{1}{2 \sin 20} \sin 40 \cos 40 \cos 80 \\
&= \frac{1}{2 \sin 20} \frac{\sin 80}{2} \cos 80 \\
&= \frac{1}{4 \sin 20} \sin 80 \cos 80 \\
&= \frac{\sin 160}{8 \sin 20} \\
&= \frac{1}{8} \text{ as } \sin 160 = \sin(180 - 20) = \sin 20
\end{aligned}$$

Problem. Find sin 18

Solution. Let $x = 18$. Then,

$$5x = 90$$

$$5x = 2x + 3x = 90$$

$$2x = 90 - 3x$$

$$\sin 2x = \sin(90 - 3x) = \cos 3x$$

$$2 \sin x \cos x = 4 \cos^3 x - 3 \cos x$$

$$2 \sin x \cos x = \cos x(4 \cos^2 x - 3)$$

$$2 \sin x \cancel{\cos x} = \cancel{\cos x}(4 \cos^2 x - 3)$$

$$2 \sin x = 4(1 - \sin^2 x) - 3$$

$$4 \sin^2 x + 2 \sin x - 1 = 0$$

$$\Rightarrow \sin x = \frac{-1 \pm \sqrt{5}}{4} \text{ or } \sin 18 = \frac{-1 + \sqrt{5}}{4}$$

Problem. Given $\cos a + \cos b = p$ and $\sin a + \sin b = q$. Find $\tan \frac{a-b}{2}$ in terms of p and q.

Solution. Note that

$$\cos(a-b) = \frac{1 - \tan^2 \frac{a-b}{2}}{1 + \tan^2 \frac{a-b}{2}}$$

Hence, its enough to find the value of $\cos(a-b)$

Squaring the given equations,
$p^2 = (\cos a + \cos b)^2 = \cos^2 a + \cos^2 b + 2\cos a \cos b$

$q^2 = (\sin a + \sin b)^2 = \sin^2 a + \sin^2 b + 2\sin a \sin b$

Now adding them,

$p^2 + q^2 = 2 + 2(\cos\cos b + \sin a \sin b)$

$\Rightarrow \cos(a-b) = \frac{p^2+q^2}{2} - 1$

Now compute, $\tan \frac{a-b}{2}$

Problem. (last problem) Prove

$$\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{5\pi}{8}\right)+\cos^2\left(\frac{7\pi}{8}\right)=2$$

Solution. Note that $\pi-\frac{\pi}{8}=\frac{7\pi}{8}$, $\pi-\frac{3\pi}{8}=\frac{5\pi}{8}$ and $\cos^2 x=\frac{1+\cos 2x}{2}$

$$\begin{aligned}
LHS &= \cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{\pi}{8}\right)\\
&= 2\left(\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)\right)\\
&= 2\left(\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}+\frac{1+\cos\left(\frac{3\pi}{4}\right)}{2}\right)\\
&= 2+\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{3\pi}{4}\right)\\
&= 2+\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)\\
&= 2
\end{aligned}$$