Week 29: Solutions (Limit of a function / continuity II)

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Note: Unless otherwise mentioned log stands for natural logarithm (the one with base e).

Solutions

1. For $a > 0$,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a \tag{1}$$

Solution: Let $v := a^x - 1$ where v is a variable dependent on x. We have,

$$\begin{aligned} a^x &= v + 1 \\ \Rightarrow x \log a &= \log(v+1) \\ \Rightarrow x &= \frac{\log(v+1)}{\log a} \end{aligned}$$

We note that as $x \to 0$, $v \to 0$

$$
\begin{aligned}
\lim_{x\to 0}\frac{a^x-1}{x} &= \lim_{v\to 0}\frac{v}{\frac{\log(v+1)}{\log a}}\\
&= \lim_{v\to 0}\frac{\log a}{\frac{1}{v}\log(v+1)}\\
&= \lim_{v\to 0}\frac{\log a}{\log(v+1)^{\frac{1}{v}}}\\
&= \frac{\log a}{\log e}\\
&= \log a
\end{aligned}
$$

external reference: http://math.stackexchange.com/a/177837

2.
$$
\lim_{x\to 0}\left(\frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)
$$

Solution: Let $y := \tan^{-1}(x)$. As $x \to 0$, $y \to 0$. The limit is rewritten as

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right) &= \lim_{y\to 0}\left(\frac{1}{\tan y}\cos^{-1}(\cos(2y))\right)\\
&= \lim_{y\to 0}\frac{2y}{\tan y}\\
&= 2
\end{aligned}
$$

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3. For $a > 0$,
$$
\lim_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}
$$

Solution:

$$
\begin{aligned}
\lim_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x} &= \lim_{x\to 0}\frac{a^{\sin x}(a^{\tan x-\sin x}-1)}{\tan x-\sin x}\\
&= \log a
\end{aligned}
$$

using these facts:
(1)
as $x \to 0$ we have $(\tan x - \sin x) \to 0$
as $x \to 0$ we have $a^{\sin x} \to 1$

4.

$$\lim_{x \to 0} \left(\frac{\log(2+x) - \log(2-x)}{x} \right)$$

<u>Solution</u>:

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{\log(2+x) - \log(2-x)}{x} \right) &= \lim_{x \to 0} \left(\frac{\log \frac{2+x}{2-x}}{x} \right) \\
&= \lim_{x \to 0} \frac{1}{x} \log \left(1 + \frac{2x}{2-x} \right) \\
&= \lim_{x \to 0} \frac{1}{x} \cdot \frac{2x}{2-x} \cdot \frac{1}{\frac{2x}{2-x}} \log \left(1 + \frac{2x}{2-x} \right) \\
&= \lim_{x \to 0} \frac{2}{2-x} \left(\frac{1}{\frac{2x}{2-x}} \log \left(1 + \frac{2x}{2-x} \right) \right) \\
&= 1
\end{aligned}$$

using the facts:
$\lim_{x \to 0} \frac{1}{x} \log(1+x) = 1$,
As $x \to 0$, $\frac{2x}{2-x} \to 0$

5.

$$\lim_{x \to 0} \frac{1 - 2^x - 5^x + 10^x}{x \sin x}$$

<u>Solution</u>:

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - 2^x - 5^x + 10^x}{x \sin x} &= \lim_{x \to 0} \frac{(2^x - 1)(5^x - 1)}{x \sin x} \\
&= \lim_{x \to 0} \frac{(2^x - 1)}{x} \cdot \frac{(5^x - 1)}{x} \cdot \frac{x}{\sin x} \\
&= \log 2 \cdot \log 5
\end{aligned}$$

6.

$$\lim_{x \to 1} \left(\tan \left(\frac{x\pi}{2} \right) \log \left(2 - \frac{1}{x} \right) \right)$$

Solution:

$$\begin{aligned}\lim_{x\to 1}\left(2-\frac{1}{x}\right)^{\tan\left(\frac{x\pi}{2}\right)} &= \lim_{x\to 1}\left(1+\left(1-\frac{1}{x}\right)\right)^{\tan\left(\frac{x\pi}{2}\right)}\\ &= \lim_{x\to 1}\left(1+\left(\frac{x-1}{x}\right)\right)^{\frac{x}{x-1}\cdot\frac{x-1}{x}\tan\left(\frac{x\pi}{2}\right)}\\ &= \lim_{x\to 1}\left(\left(1+\left(\frac{x-1}{x}\right)\right)^{\frac{x}{x-1}}\right)^{\frac{x-1}{x}\tan\left(\frac{x\pi}{2}\right)}\\ &= e^{\frac{-2}{\pi}}\end{aligned}$$

Hence,

$$\lim_{x\to 1}\left(\tan\left(\frac{x\pi}{2}\right)\log\left(2-\frac{1}{x}\right)\right)=\frac{-2}{\pi}$$

Using the following facts:
$\lim_{x\to 0}\frac{1}{x}\log(1+x)=1$
As $x\to 1$, $\frac{x-1}{x}\to 0$

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{x}\tan\left(\frac{x\pi}{2}\right) &= \lim_{y\to 0}\frac{-y}{1-y}\tan\left(\frac{\pi}{2}(1-y)\right) \qquad \text{where } y=1-x\\ &= \lim_{y\to 0}\frac{-y}{1-y}\cot\left(\frac{\pi y}{2}\right)\\ &= \lim_{y\to 0}\frac{-1}{1-y}\frac{y}{\tan\left(\frac{\pi y}{2}\right)}\\ &= \lim_{y\to 0}\frac{-1}{1-y}\frac{\frac{\pi y}{2}}{\tan\left(\frac{\pi y}{2}\right)}\cdot\frac{2}{\pi}\\ &= \frac{-2}{\pi}\end{aligned}$$

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7. Find a and b provided:

$$\lim_{x\to 0}(1+ax+bx^2)^{\frac{1}{x}}=e^3$$

Solution:

$$\begin{aligned}\lim_{x\to 0}(1+ax+bx^2)^{\frac{1}{x}} &= \lim_{x\to 0}\left((1+x(a+bx))^{\frac{1}{x(a+bx)}}\right)^{a+bx}\\ &= e^a\end{aligned}$$

Hence, $a=3$ and b can be any real number.

8. Let n be a positive integer. Find,

$$\lim_{x\to\frac{\pi}{2}}\left(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x}\right)^{\cos^2 x}$$

Solution: Let

$$E=\left(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x}\right)^{\cos^2 x}$$

We notice that $1\le\sec^2 x<\infty$. Hence, $r^{\sec^2 x}<n^{\sec^2 x}\quad\forall\, r=1,2,\ldots,n-1$. This gives,

$$\begin{aligned}\left(n^{\sec^2 x}\right)^{\cos^2 x}<&E<\left(n^{\sec^2 x}+n^{\sec^2 x}+\cdots+n^{\sec^2 x}\right)^{\cos^2 x}\\ n<&E<\left(n\cdot n^{\sec^2 x}\right)^{\cos^2 x}\end{aligned}$$

Applying limit allover,

$$\begin{aligned}\lim_{x\to 0} n<\lim_{x\to 0}\left(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x}\right)^{\cos^2 x}&<\lim_{x\to 0}\left(n\cdot n^{\sec^2 x}\right)^{\cos^2 x}\\ n\le\lim_{x\to 0}\left(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x}\right)^{\cos^2 x}&\le 1\cdot n\end{aligned}$$

Hence,

$$\lim_{x\to\frac{\pi}{2}}\left(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x}\right)^{\cos^2 x}=n$$

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9. Let $i(x)$ denote the integral part of a real number x. Find

$$\lim_{n\to\infty}\frac{i(x)+i(2x)+\cdots+i(nx)}{n^2}$$

where x is some real number.

Solution: Note that $kx-1<i(kx)<kx+1\quad\forall\, k\in\mathbb{N}$. Adding such equations for $k=1$ to $k=n$, we get

$$\left(x\sum_{k=1}^{n}k\right)-n<\sum_{k=1}^{n}i(kx)<\left(x\sum_{k=1}^{n}k\right)+n$$

On dividing by n^2 allthrough,

$$\left(x\frac{n(n+1)}{2n^2}\right)-\frac{1}{n}<\sum_{k=1}^{n}i(kx)<\left(x\frac{n(n+1)}{2n^2}\right)+\frac{1}{n}$$

On applying limit as $n \to \infty$ allthrough,

$$\frac{x}{2} < \lim_{n\to\infty} \left(\sum_{k=1}^{n} i(kx) \right) < \frac{x}{2}$$

gives the desired limit.

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10. Let

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 + a_1 \\ a_3 &= 1 + a_1 a_2 \\ &\dots \\ a_n &= 1 + a_1 a_2 \dots a_{n-1} \end{aligned}$$

Find

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

<u>Solution</u>: Note that $a_n \geq a_{n-1} + 1 \quad \forall n \geq 2,\ n \in \mathbb{N}$. Hence the sequence a_n diverges to ∞.

$$\begin{aligned} \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} &= 2 - \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \\ &= 2 - \left(\frac{-a_2 + a_1}{a_1 a_2} \right) + \frac{1}{a_3} + \dots + \frac{1}{a_n} \\ &= 2 - \left(\frac{-1}{a_1 a_2} \right) + \frac{1}{a_3} + \dots + \frac{1}{a_n} \\ &= 2 - \left(\frac{-1}{a_1 a_2} - \frac{1}{a_3} \right) + \dots + \frac{1}{a_n} \\ &= 2 - \left(\frac{-a_3 + a_1 a_2}{a_1 a_2 a_3} \right) + \frac{1}{a_4} \dots + \frac{1}{a_n} \\ &= 2 - \left(\frac{-1}{a_1 a_2 a_3} - \frac{1}{a_4} \right) \dots + \frac{1}{a_n} \\ &= \dots \\ &= 2 + \frac{1}{a_1 a_2 \dots a_n} \end{aligned}$$

As $n \to \infty, a_1 a_2 \dots a_n \to \infty$. Thus, $\sum_{n=1}^{\infty} \frac{1}{a_n} = 2$

11. Let

$$f(x) = \lim_{n\to\infty} \frac{1 - x^{2n}}{1 + x^{2n}}$$

Find the points of discontinuity of the function.

Solution: We know that

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0 & \text{if } \mid x \mid < 1 \\ \infty & \text{if } \mid x \mid > 1 \end{cases}$$

Hence,

$$f(x) = \lim_{n\to\infty} \frac{1 - x^{2n}}{1 + x^{2n}} = \begin{cases} 1 & \text{if } \mid x \mid < 1 \\ -1 & \text{if } \mid x \mid > 1 \end{cases}$$

The function is discontinuous at $x = 1$ and $x = -1$

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12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that the $f(x) = x$ does not have a solution. Does $f(f(x)) = x$ have a real solution?

Solution: Let $g(x) := f(x) - x$. Then, $g(x)$ is continuous. Since, $g(x)$ is never 0, $g(x)$ is always positive or always negative (by intermediate value theorem). Consider,

$$\begin{aligned} f(f(x)) - x &= f(f(x)) - f(x) + f(x) - x \\ &= \underbrace{f(f(x)) - f(x)}_{g(f(x))} + \underbrace{f(x) - x}_{g(x)} \end{aligned}$$

The expression can never be zero as both terms are either positive or both are negative.
