Week 28: Problems (Limit of a function and continuity I)

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1 Main ideas

1. A function $f(x)$ is said to be continuous at $x = a \in \mathbb{R}$ if for every sequence $x_1, x_2, \ldots$ converging to a, we have the sequence $f(x_1), f(x_2), \ldots$ converging to $f(a)$.

   Exercise: The function

   $$f(x) = \begin{cases} 0 \text{ if x is irrational} \\ 1 \text{ if x is rational} \end{cases}$$

   is not continuous. Consider two sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n\sqrt{2}}$. The former sequence is made up of rational numbers and the latter with irrationals. Both converge to zero as n grows large. But $f(x_n) \to 1$ and $f(y_n) \to 0$. If the function was continuous, then both $f(x_n)$ and $f(y_n)$ should have both converged to the same value as $f(0)$ which is 1, as 0 is rational.

   Note: I am avoiding the $\epsilon - \delta$ definition of continuity on purpose. Our discussion is focussed mainly on limits rather than discussion of continuity.

2. An equivalent way proving(disproving) continuity at a point is by establishing left handed and right handed limits.

Let L be the set of all monotonically increasing sequences converging to a. If $\exists\,\ell \in \mathbb{R}$ such that $f(x_n) \to \ell \quad \forall\, x_n \in L$, we define the left-handed limit of the function at $x = a$ as

$$\lim_{x \to a-} f(x) = \ell$$

Similarly, we define a right handed limit at $x = a$.

Exercise: Complete the description of the right handed limit:

$$\lim_{x \to a+} f(x)$$

Exercise: Is the function continuous:

$$f(x) = \begin{cases} x + 1 \text{ ,if } x > 0 \\ e^x \text{ ,if } x \leq 0 \end{cases}$$

3.
$$\lim_{n \to \infty, n \in \mathbb{N}} \left(1 + \frac{1}{n}\right)^n = e$$

where e denotes an irrational number that lies between 2 and 3.

4. We may extent:
$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

where it is implicit that x is real variable.

5. We may rewrite the previous limit as

$$\lim_{y \to 0+} (1 + y)^{\frac{1}{y}} = e$$

6.
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

and
$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

Note that $\sin x < x < \tan x$ is true for all $x \in (0, \frac{\pi}{2})$.

2 Problems

1.
$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=\frac{1}{e}$$

2.
$$\lim_{x\to 0}\frac{\sin\lfloor x\rfloor}{\lfloor x\rfloor}$$

3.
$$\lim_{x\to a}\frac{\tan x-\tan a}{x-a}$$

4.
$$\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$$

5.
$$\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$$

6.
$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$$

7. If $f(x)=min\{x,x^2\}$, then find
$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

8.
$$\lim_{x\to 1}\frac{1+\cos\pi x}{\tan^2\pi x}$$

9.
$$\lim_{x\to\pi/2}(\sec x-\tan x)$$

10.
$$\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x}$$

11.
$$\lim_{x\to 0}\cos x^{\cot x}$$

12. Find
$$\sum_{n=1}^{\infty}\frac{n^2}{n!}$$
