Week 27: Problems (limit of a sequence II)

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1 Main ideas

1. **Cauchy's criterion for convergence:** A sequence a_n is convergent iff
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $| a_n - a_{n+k} | < \varepsilon \quad \forall n \geq n_0$ and $\forall k \in \mathbb{N}$
2. A bounded monotonic sequence is convergent.
3. Techniques of solving a recursively defined sequence.
4. Every sequence has a convergent subsequence. (**Bolzano-Weierstrass**)
5. $$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$
where is an irrational number between 2 and 3.

2 Problems

1. Let $a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$. Prove that the sequence a_n converges.

2. Is it possible to have a divergent sequence a_n with the property:
   $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\mid a_n - a_{n+1} \mid < \varepsilon \quad \forall n \geq n_0$

3. Let
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} \cdots + \frac{1}{n+n}$$
   and
$$b_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$$
   Show that both sequences converge to same limit.(hint: both are monotonic)

4. Prove that
$$a_n = \left(1 + \frac{1}{n}\right)^n$$
   and
$$b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$
   converge to a same limit.

5. Prove that
$$\left(1 + \frac{1}{n^2}\right)^n \to 1$$

6. Let $a_1 = 0$ and
$$a_{n+1} = \frac{3a_n + 1}{a_n + 3} \quad \forall n \geq 1$$
   Prove that a_n is converges. Find the limit.

   (hint: a_n is monotonic and bounded)

7. Prove that
$$b_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$
   diverges.

8. Prove that
$$c_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$
   converges to a value between $\frac{1}{2}$ and 1.
