Week 26: Solutions
(limit of a sequence I)

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1 Solutions

1. *Limit of convergence of a sequence(if convergent) is unique.*

   Solution: Let l_1 and l_2 be two distinct limits of a sequence a_n. Let $k = \mid l_1 - l_2 \mid$. Consider neighborhoods(open interval) of length $\frac{k}{2}$ around l_1 and l_2. The neighborhoods do not intersect. Hence, it is impossible for the sequence to be in both the neighborhoods eventually.

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2. *A convergent sequence is bounded.*

   Solution: Suppose the sequence converges to the limit l. Lets take $\varepsilon = 1$ (we could have chosen any other positive constant too). Then, $\exists\, n_0 \in \mathbb{N}$ such that $\mid a_n - l \mid < 1 \quad \forall\, n \geq n_0$. Hence, eventually the sequence lies in $(l-1, l+1)$.

   Among the initial $a_1, a_2, \ldots, a_{n_0-1}$ terms: Let $m = min\{a_1, a_2, \ldots, a_{n_0-1}\}$ and

$M = max\{a_1, a_2, \ldots, a_{n_0-1}\}$. Hence, the first $n_0 - 1$ terms lie in the interval $[m, M]$.

Hence the sequence is bounded between $[min\{m, l-1\}, max\{M, l+1\}]$

(In essence, sequence is bounded for n_0th term onwards. Before that we have finitely many terms, hence the sequence is bounded there too.)

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3. *Suppose* $a_n \geq 0 \quad \forall n \in \mathbb{N}$. *Then,* $a_n \to \ell \iff \sqrt{a_n} \to \sqrt{\ell}$

<u>Solution</u>: $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $| a_n - \ell | < \varepsilon \quad \forall n \geq n_0$. Hence,

$$| a_n - \ell | = | \left(\sqrt{a_n} - \sqrt{\ell}\right) \left(\sqrt{a_n} + \sqrt{\ell}\right) | < \varepsilon$$
$$\Rightarrow | \left(\sqrt{a_n} - \sqrt{\ell}\right) | < \frac{\varepsilon}{\left(\sqrt{a_n} + \sqrt{\ell}\right)}$$

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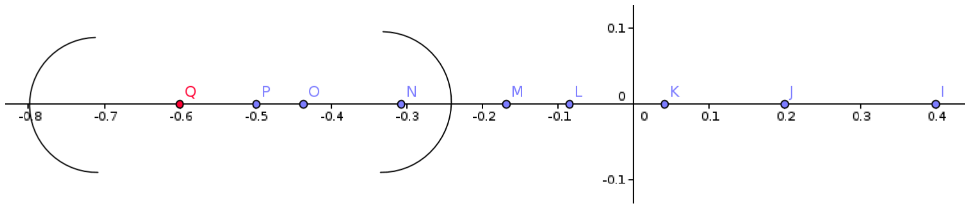
4. *A sequence of positive numbers cannot converge to a negative number (but it may converge to zero).*

<u>Solution</u>: Suppose a sequence of positive terms a_n converges to a negative number g. There exists an open interval containing g but not 0 (an open interval entirely on the left side of 0 on the number line). Eventually, the sequence lies in the open interval as it converges to g. This would mean that the terms of the sequence eventually are negative. This is a contradiction to the fact that a_n is a sequence with positive terms.



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5. *Prove that* $a_n \to \ell \Rightarrow | a_n | \to | \ell |$. *Is the converse true?*

Solution: Note the inequality, $|| a | - | b || \leq | a-b |$ is true $\forall\, a, b \in \mathbb{R}$. Hence, $\forall\, \varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $|| a_n | - | \ell || \leq | a_n - \ell | < \varepsilon \quad \forall\, n \geq n_0$

Converse is false, consider $a_n = (-1)^n$

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6. *Suppose $a_n \leq b_n \leq c_n \quad \forall\, n \in \mathbb{N}$. If $a_n \to \ell$ and $c_n \to \ell$, then $b_n \to \ell$*

   Solution: We need to notice that $| a_n - \ell | < \varepsilon$ and $| c_n - \ell | < \varepsilon$ along with $a_n \leq b_n \leq c_n$ implies that $| b_n - \ell | < \varepsilon$

   This is also known as *sandwich principle*

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7. *Use the previous fact or otherwise prove that*

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1$$

   Solution: Let $E = \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right)$

   Observe that $\frac{1}{\sqrt{n^2+1}}$ is the largest and $\frac{1}{\sqrt{n^2+n}}$ is the smallest among all the terms.

   Note that

$$\left(\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) < E < \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}} \right)$$

$$\Rightarrow \frac{n}{\sqrt{n^2+n}} < E < \frac{n}{\sqrt{n^2+1}}$$

   Note that as $n \to \infty$, we have

$$\frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \to 1$$

   and

$$\frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to 1$$

   By sandwich principle, $\lim_{n\to\infty} E = 1$

8.
$$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{if } \mid r \mid < 1 \\ \text{unbounded on the positive side} & \text{if } r > 1 \\ \text{undefined} & \text{if } r < 1 \end{cases}$$

Solution: Case when $r = 1$ is trivial. If $\mid r \mid < 1$, we have $(\mid r \mid)^2 < \mid r \mid$. Hence, the sequence $(\mid r \mid)^n$ steadily decreases. Since, it can never converge to a non-negative limit we suspect that it tends to zero. We prove that the sequence r^n converges to 0 for $0 < r < 1$. The case when $-1 < r < 0$ is oscillatory, but r^n converges to 0. (this is because $\mid r^n \mid$ is strictly decreasing)

Suppose r^n converges to $\ell \neq 0$ (where $0 < r < 1$). For some $0 < \varepsilon < \frac{\ell(1-r)}{r}$, $\exists\, n_0 \in \mathbb{N}$ such that $r^n - \ell < \varepsilon \quad \forall n \geq n_0$.

$$\begin{aligned} r^n &< \ell + \varepsilon \\ r^{n+1} &< r\,(\ell + \varepsilon) \\ &< r\left(\ell + \frac{\ell(1-r)}{r}\right) \\ &= \ell \end{aligned}$$

$r^{n+1} < \ell$ is indicative of the fact that ℓ cannot be the limit. Hence, it rules out existence any positive limit.

When $r > 1$, we may set $r' = \frac{1}{r}$. By the previous argument, $(r')^n \to 0$, hence, $r^n \to \infty$ as r is positive.

When $r < -1$, the $\mid r \mid \to \infty$, but r being negative leads to an alternating divergent sequence. Hence, the limit fails to exist.

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9. Prove that if $s_n \to \ell$, then
$$\frac{s_1 + s_2 + \cdots + s_n}{n} \to \ell$$

Solution: Note that $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\ell - \varepsilon < s_n < \ell + \varepsilon \quad \forall n \geq n_0$. For some $n > n_0$,

$$\frac{s_1 + s_2 + \cdots + s_n}{n} = \frac{s_1 + s_2 + \cdots + s_{n_0 - 1}}{n} + \frac{s_{n_0} + \cdots + s_n}{n}$$

We note that,

$$\frac{(n - n_0 + 1)(\ell - \varepsilon)}{n} < \frac{s_{n_0} + \cdots + s_n}{n} < \frac{(n - n_0 + 1)(\ell + \varepsilon)}{n}$$

As $n \to \infty$, we have

$$\frac{(n - n_0 + 1)(\ell \pm \varepsilon)}{n} \to \ell \pm \varepsilon \tag{1}$$

$$\frac{s_1 + s_2 + \cdots + s_{n_0 - 1}}{n} \to 0 \tag{2}$$

$$\frac{s_{n_0} + \cdots + s_n}{n} \to \ell \tag{3}$$

due to (1) and sandwich principle.

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10. *Let* $t_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$. *Is it possible that* s_n *does not converge but* t_n *does?*

    Solution: Yes, consider $s_n = (-1)^n$

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11. *Find (if it exists)*

$$\lim_{n \to \infty} \sqrt{n^2 + n} - n$$

Solution:

$$\begin{aligned}\lim_{n\to\infty}\sqrt{n^2+n}-n &= \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}\\ &= \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}\\ &= \frac{1}{2}\end{aligned}$$

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12. *Prove that if $x > 0$ (note that x is not the variable), then*

$$x^{\frac{1}{n}} \to 1$$

Solution: Suppose $x > 1$. Then, we claim that $x^{\frac{1}{n}} > 1 \quad \forall n \in \mathbb{N}$. If $x^{\frac{1}{n}} \leq 1$, we have $x \leq 1$ (on raising to power n on both sides), thus contradicting our assumption. Hence, let $s_n := x^{\frac{1}{n}} - 1$. Proving $s_n \to 0$ would be equivalent to proving $x^{\frac{1}{n}} \to 1$. (Note that s_n is a sequence with positive terms).

$$\begin{aligned}(1+s_n)^n &\geq n\cdot s_n > 0 \quad \text{(on applying binomial theorem)}\\ \Rightarrow x &\geq n\cdot s_n > 0\\ \Rightarrow \frac{x}{n} &\geq s_n > 0\end{aligned}$$

Applying sandwich theorem as $n \to \infty$, we get $s_n \to 0$. Equivalently, $x^{\frac{1}{n}} \to 1$.

For $0 < x < 1$, let $y := \frac{1}{x}$

Then, $y > 1$ and $y^{\frac{1}{n}} = \left(\frac{1}{x}\right)^{\frac{1}{n}} = \left(\frac{1}{x^{\frac{1}{n}}}\right) \to 1$

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13. Prove that

$$n^{\frac{1}{n}} \to 1$$

<u>Solution</u>: We claim that $n^{\frac{1}{n}} > 1 \quad \forall n \in \mathbb{N}$ and $n > 1$. If $n^{\frac{1}{n}} \leq 1$, we have $n \leq 1$ (on raising to power n on both sides). Hence, let $s_n := n^{\frac{1}{n}} - 1$. Proving $s_n \to 0$ would be equivalent to proving $n^{\frac{1}{n}} \to 1$. (Note that s_n is a sequence with positive terms).

$$\begin{aligned} (1+s_n)^n &\geq \binom{n}{2}(s_n)^2 > 0 \quad \text{(on applying binomial theorem)} \\ \Rightarrow n &\geq \frac{n(n-1)}{2}(s_n)^2 > 0 \\ \Rightarrow \frac{2}{n-1} &\geq (s_n)^2 > 0 \end{aligned}$$

Applying sandwich theorem as $n \to \infty$, we get $(s_n)^2 \to 0$. Using Problem 3, $s_n \to 0$. Equivalently, $n^{\frac{1}{n}} \to 1$.

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14. Find (if it exists)

$$\lim_{n\to\infty} \frac{n - \cos n}{n}$$

<u>Solution</u>: Since, $-1 \leq \cos x \leq 1$, $\frac{\cos n}{n} \to 0$ as $n \to \infty$ (although its is oscillatory). Hence the required limit is 1.

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