Week 26: Problems

(limit of a sequence I)

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1 Main ideas

1. A sequence(infinite) denotes by a_n where $n \in \mathbb{N}$ is the ordered tuple $a_1, a_2, a_3, \ldots$

2. A sequence a_n is said to converge to $\ell \in \mathbb{R}$ if

$$\forall\, \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \text{ such that}$$
$$|\, a_n - \ell \,| < \varepsilon \quad \forall\, n \geq n_0$$

   Symbolically we write, $a_n \to \ell$ as $n \to \infty$ (read as: a_n tends to ℓ, as n grows large)

   Sometimes, we further shorten it and simply write $a_n \to \ell$

   examples: $a_n = \frac{1}{n}, b_n = \frac{n}{n+1}, c_n = \frac{(-1)^n}{n}$

3. A sequence that is not convergent is said to be a divergent sequence. A divergent sequence may be oscillating or it may be unbounded either on positive

or the negative side.

examples: $d_n = (-1)^n, e_n = \sin(n), f_n = -n$

4. The statement "$a_n \to \ell$ as $n \to \infty$" is equivalent to

$$\lim_{n\to\infty} a_n = \ell$$

(Read as: Limit of the sequence a_n as n grows large is ℓ).
Although, we do not mention explicitly, n grows large only along positive integer values.)

5. If the sequence is divergent, the limit of the sequence may not exist. For example, consider $d_n = (-1)^n$

6. Suppose $a_n \to a$ and $b_n \to b$.Then,

$$a_n + b_n \to a + b \tag{1}$$

$$a_n b_n \to ab \tag{2}$$

$$\frac{a_n}{b_n} \to \frac{a}{b} \qquad \text{where } b \neq 0 \tag{3}$$

2 Problems

1. Limit of convergence of a sequence(if convergent) is unique.

2. A convergent sequence is bounded.

3. Suppose $a_n \geq 0 \quad \forall n \in \mathbb{N}$. Then, $a_n \to \ell \Longleftrightarrow \sqrt{a_n} \to \sqrt{\ell}$

4. A sequence of positive numbers cannot converge to a negative number (but it may converge to zero).

5. Prove that $a_n \to \ell \Rightarrow | a_n | \to | \ell |$. IS the converse true?

6. Suppose $a_n \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$. If $a_n \to \ell$ and $c_n \to \ell$, then $b_n \to \ell$

7. Use the previous fact or otherwise prove that

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1$$

8.

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{if } \mid r \mid < 1 \\ \infty & \text{if } r > 1 \\ \text{undefined} & \text{if } r \leq -1 \end{cases}$$

9. Prove that if $s_n \to \ell$, then

$$\frac{s_1 + s_2 + \cdots + s_n}{n} \to \ell$$

10. Let $t_n = \frac{s_1+s_2+\cdots+s_n}{n}$. Is it possible that s_n does not converge but t_n does?

11. Find (if it exists)

$$\lim_{n\to\infty} \sqrt{n^2+n} - n$$

12. Prove that if $x > 0$ (note that x is not the variable), then

$$x^{\frac{1}{n}} \to 1$$

13. Prove that

$$n^{\frac{1}{n}} \to 1$$

14. Find (if it exists)

$$\lim_{n\to\infty} \frac{n - \cos n}{n}$$
