Lecture 23: More properties of triangles

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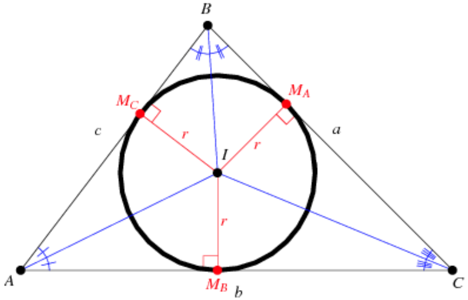
Contents

Fact

For any $\triangle ABC$, there exists a circle in the interior of the triangle touching(at exactly one point) all the three edges.

This circle is unique, it is called the incircle, its centre is called incentre denoted by I and its radius is called the inradius denoted by r.

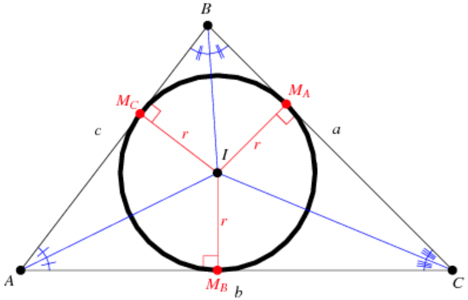
Exercise. Prove the above fact.



Solution. Recall the fact that angular bisectors are concurrent. We have $\triangle AM_cI \equiv \triangle AM_bI$ by SAS postulate. Hence, $IM_c = IM_b$. By similar arguments, $IM_c = IM_b = IM_a$.

Result

Prove that $r = \frac{\Delta}{s}$



Solution.

$$\begin{aligned}\Delta &= Area(\triangle ABC)\\ &= Area(\triangle AIB) + Area(\triangle BIC) + Area(\triangle CIA)\\ &= \frac{cr}{2} + \frac{ar}{2} + \frac{br}{2}\\ &= r\left(\frac{a+b+c}{2}\right)\\ &= rs\end{aligned}$$

Result

Prove $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{b}{2} = (s-c)\tan\frac{C}{2}$

Solution.

$$\begin{aligned} r &= \frac{\Delta}{s} \\ &= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \\ &= (s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ &= (s-a)\tan\frac{A}{2} \end{aligned}$$

Exercise. Prove that above fact geometrically
(by proving that $AM_b = s-a$)

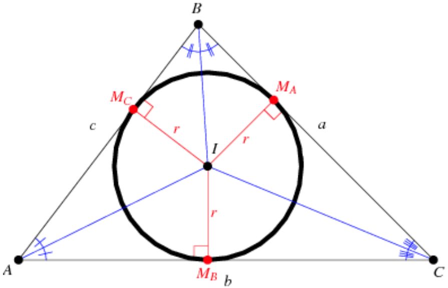
Result

(i)

$$r = a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

(ii)

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$



Solution.

$$a = r\cot\frac{B}{2} + r\cot\frac{C}{2}$$
$$\Rightarrow r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}}$$
$$= \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\left(\frac{B+C}{2}\right)}$$
$$= \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\left(\frac{A}{2}\right)}$$

Solution.

$$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\left(\frac{A}{2}\right)}$$
$$= \frac{2R\sin A\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\left(\frac{A}{2}\right)}$$
$$= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Exercise.

Prove

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$$

Solution.

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{a+b+c}{abc}=\frac{2s}{abc}=\frac{\frac{2s}{4\Delta}}{\frac{abc}{4\Delta}}=\frac{1}{2Rr}$$

Exercise.

Prove that $R \geq 2r$. When does equality occur?

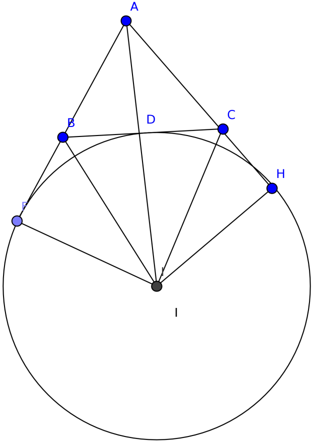
Solution. Equivalently, we need to prove

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$$

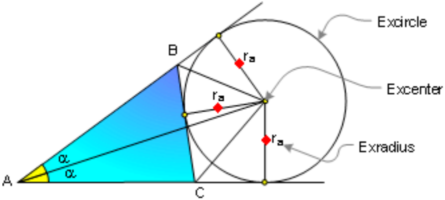
The LHS maximizes when $\sin\frac{A}{2}=\sin\frac{B}{2}=\sin\frac{C}{2}$ (by AM-GM inequality). Equality occurs when $A=B=C=\frac{\pi}{3}$.

Fact

In any $\triangle ABC$, angular bisector of A, angular bisectors of exterior angles of B and C are concurrent.



Solution. We have $|IF| = |ID|$ and $|IH| = |ID|$. This would mean that IA is the angular bisector of A.



The circle in the picture is called the excircle opposite to A. Its radius is denoted by r_a.

Result

$$r_a = \frac{\Delta}{s-a}, r_b = \frac{\Delta}{s-b}, r_c = \frac{\Delta}{s-c}$$

Solution.

$Area(\triangle ABC) = Area(\triangle ABI) + Area(\triangle ACI) - Area(\triangle IBC)$

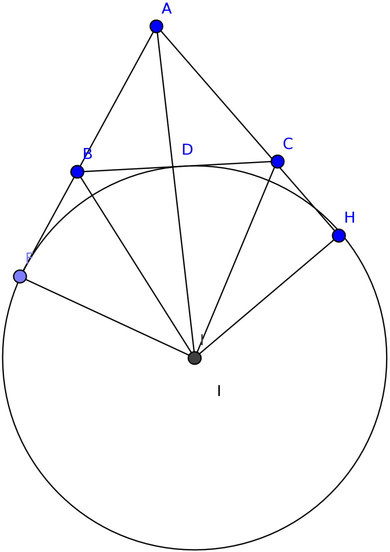
$$\Delta = \frac{|AB||IF|}{2} + \frac{|AC||IH|}{2} - \frac{|BC||ID|}{2}$$

$$= \frac{cr_a}{2} + \frac{br_a}{2} - \frac{ar_a}{2}$$

$$= (c+b-a)\frac{r_a}{2}$$

$$= (2s-2a)\frac{r_a}{2}$$

$$= (s-a)r_a$$



Result

$$r_a = s \tan \frac{A}{2}, r_b = s \tan \frac{B}{2}, r_c = s \tan \frac{C}{2}$$

Solution.

$$\begin{aligned} r_a &= \frac{\Delta}{s-a} \\ &= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a} \\ &= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ &= s \tan \frac{A}{2} \end{aligned}$$

Exercise. Prove the above result geometrically.
(by proving $\mid AF \mid = \mid AH \mid = s$)

Result

$$r_a = \frac{a \cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, r_b = \frac{b \cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}, r_c = \frac{c \cos\frac{B}{2}\cos\frac{A}{2}}{\cos\frac{C}{2}}$$

hint: $a = BD + DC = r\left(\cot(\angle IBD) + \cot(\angle ICD)\right)$

Result

$$r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Hint: Expand the previous result with $a = 2R\sin A$

Exercise.

Prove

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$$

Solution.

$$\frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{3s-(a+b+c)}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

Exercise.

$$rr_ar_br_c = \Delta^2$$

Solution.

$$LHS = \frac{\Delta}{s}\frac{\Delta}{s-a}\frac{\Delta}{s-b}\frac{\Delta}{s-c} = \frac{\Delta^4}{\Delta^2}$$

Exercise.

In $\triangle ABC, r_a = 8, r_b = 12, r_c = 24$. Find the length of the sides of the triangle.

Hint: We have

$$s-a = \frac{\Delta}{8}, s-b = \frac{\Delta}{12}, s-c = \frac{\Delta}{24}$$

Find s in terms of Δ. This leads to r in terms of Δ. After this use, the previous exercise.