Week 21: Solutions

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course webpage: http://bit.ly/trig2013

Solutions to Week 21 problems

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1. $$\frac{b-c}{b+c} = \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}$$

$$= \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}$$

$$= \frac{\tan\frac{B-C}{2}}{\tan\left(\frac{\pi}{2}-\frac{A}{2}\right)} = \frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}$$

The result is known as 'tan rule' or 'Napier's analogy'.

2. $$\frac{b-c}{a} = \frac{\sin B - \sin C}{\sin A}$$ (on the same lines as prev problem)

$$\Rightarrow \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\cos\frac{B+C}{2}\sin\frac{B+C}{2}}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

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3. $$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\sin\frac{B+C}{2}}$$

$$= \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$$

4. $\sum_{cyc} (a-b)\sin C$

$= a\sin C - b\sin C + b\sin A - c\sin A + c\sin B - a\sin B$

$= (a\sin C - c\sin A) + (b\sin A - a\sin B) + (c\sin B - b\sin C)$

$= 0 + 0 + 0 = 0$ (by sine rule).

<u>Aliter</u>: put $\sin C = \dfrac{c}{2R}$ and proceed.

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5. $\sum_{cyc} (a+b)\cos C$

$= (a+b)\cos C + (b+c)\cos A + (c+a)\cos B$

$= (a\cos C + c\cos A) + (b\cos C + c\cos B) + (a\cos B + b\cos A)$

$= b + a + c$

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6. $(a+b-c)(b+c-a) = 3ac$

$\Rightarrow (2s-2c)(2s-2a) = 3ac$

$\Rightarrow \dfrac{(s-c)(s-a)}{ac} = \dfrac{3}{4}$

$\Rightarrow \sin^2\left(\dfrac{B}{2}\right) = \dfrac{3}{4} \Rightarrow \sin\left(\dfrac{B}{2}\right) = \pm\dfrac{\sqrt{3}}{2}$

$\Rightarrow B = 2\pi/3$ as negative angle is not possible.

7. $\tan\left(\frac{A}{2}\right) = \frac{5}{6}$, $\tan\left(\frac{C}{2}\right) = \frac{2}{5}$

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{5}{6} \text{ and } \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \frac{2}{5}$$

Taking their products

$$\frac{s-b}{s} = \frac{1}{3} \Rightarrow 3s - 3b = s$$

$$\Rightarrow 2s = 3b \Rightarrow a+b+c = 3b$$

$$\Rightarrow a + c = 2b$$

Hence, a, b, c are in AP.

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8. $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$ in terms of 's'.

$$= b\,\frac{s(s-c)}{ba} + c\,\frac{s(s-b)}{ac}$$

$$= s\left[\frac{s-c}{a} + \frac{s-b}{a}\right] = s\left[\frac{2s-b-c}{a}\right]$$

$$= s \times \frac{a}{a} = s$$

9. $\sum_{cyc} a \sin(B-C) = \sum_{cyc} 2R \sin A \sin(B-C)$

$$= 2R \sum_{cyc} \sin(B+C) \sin(B-C)$$

$$= 2R \sum_{cyc} \frac{\cos 2C - \cos 2B}{2}$$

$$= R \sum_{cyc} \cos(2C) - \cos(2B)$$

$$= 0$$

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10. $\cot A + \cot B + \cot C$

$$= \sum_{cyc} \frac{\cos A}{\sin A} = \sum_{cyc} \frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{2\Delta}{bc}}$$

$$= \sum_{cyc} \frac{b^2+c^2-a^2}{4\Delta}$$

$$= \frac{a^2+b^2+c^2}{4\Delta}$$

11. $8R^2 = a^2+b^2+c^2$

$$\Rightarrow 8R^2 = \sum_{cyc} (2R\sin A)^2$$

$$= 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$$

$$\Rightarrow 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$2 = 3 - (\cos^2 A + \cos^2 B + \cos^2 C)$$

$$\Rightarrow 1 = \cos^2 A + \cos^2 B + \cos^2 C$$

$$\Leftrightarrow 1 = \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$$

$$\Leftrightarrow -\frac{1}{2} = \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C)$$

$$\Leftrightarrow -1 = \cos 2A + \cos 2B + \cos 2C$$

$$RHS = 2\cos(A+B)\cos(A-B) + \cos 2(\pi-(A+B))$$

$$= 2\cos(A+B)\cos(A-B) + 2\cos^2(A+B) - 1$$

$$= -1 + 2\cos(A+B)[\cos(A-B) + \cos(A+B)]$$

$$= -1 + 2\cos C \quad \cos A \cos B$$

$$\Rightarrow \cos A \cos B \cos C = 0$$

One of them is zero, $\Rightarrow$ there is an right angle.

12. $$\sum_{cyc} \frac{\cos A}{a} = \sum_{cyc} \frac{\frac{b^2+c^2-a^2}{2bc}}{a}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

$$RHS = \frac{a^2+b^2}{abc}$$

$LHS = RHS \Rightarrow 2(a^2+b^2) = a^2+b^2+c^2$

$\Rightarrow c^2 = a^2+b^2$

Hence, $\angle C = \pi/2$

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16. $$(a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)$$

$$= 2s\left(1 - \tan\frac{A}{2}\tan\frac{B}{2}\right)\tan\left(\frac{A+B}{2}\right)$$

... from $\tan\frac{(A+B)}{2}$ formula.

$$= 2s\left(1 - \sqrt{\frac{(s-c)(s-b)}{s(s-a)}} \times \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right)\cot\left(\frac{C}{2}\right)$$

as $\tan\frac{A+B}{2}$

$= \tan\left(\frac{\pi - C}{2}\right)$

$$= 2s\left(1 - \frac{s-c}{s}\right)\cot\frac{C}{2}$$

$$= 2c\cot\frac{C}{2}$$

17. Note that $\cot A = \dfrac{\cos A}{\sin A}$

$$= \frac{\dfrac{b^2+c^2-a^2}{2bc}}{\dfrac{2\Delta}{bc}} = \frac{b^2+c^2-a^2}{4\Delta}$$

Hence, it would be equivalent to prove that

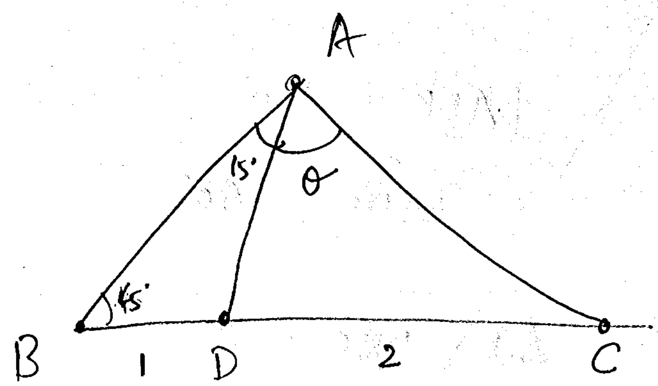
$b^2+c^2-a^2,\quad a^2+c^2-b^2,\quad a^2+b^2-c^2$ are in AP

$\Leftrightarrow (a^2+b^2+c^2)-2a^2,\quad (a^2+b^2+c^2)-2b^2,\quad (a^2+b^2+c^2)-2c^2$ are in AP.

$\Leftrightarrow -2a^2,\quad -2b^2,\quad -2c^2$ are in AP.

$\Leftrightarrow a^2,\quad b^2,\quad c^2$ are in AP.

13.



From the fig. $\angle ADB = 120°$

$\Rightarrow \angle ADC = 60°$

Let $\angle DAC = \theta$

$$\frac{CD}{\sin\theta} = \frac{AC}{\sin 60°}$$ --- on applying Sine rule in ΔADC

$$\frac{BC}{\sin(\theta+15)} = \frac{AC}{\sin 45°}$$ -- on applying Sine rule in ΔABC

Eliminating AC from the two equations,

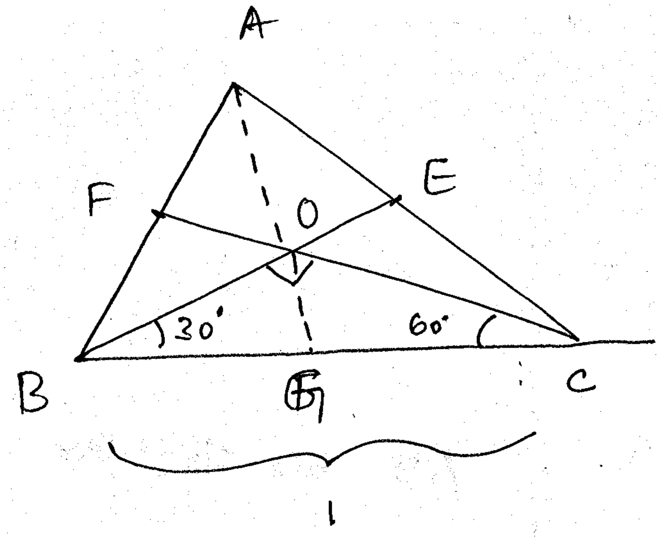
$$\frac{CD}{BC} = \frac{2}{3} = \frac{\sin\theta \sin 45°}{\sin(\theta+15°)\sin 60°}$$

But note that, $\left(\frac{\sin 45°}{\sin 60}\right)^2 = \frac{2}{3}$

$\Rightarrow \theta = 45°$ is a solution.

$\Rightarrow \angle C = 75°$.

14. BE and CF are medians. Let the medians meet at O. (centroid)



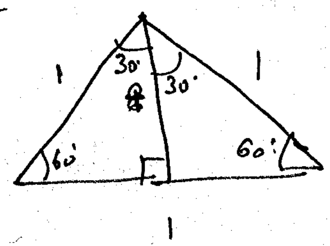
We have,

$$\angle BOC = \frac{\pi}{2} \quad \text{and}$$

AG is the median as medians in any triangle are concurrent.

Not that medians divide the triangle into six equal parts.

$$\text{Area of } \Delta BOC = \frac{\sqrt{3}}{4 \times 2} = \frac{\sqrt{3}}{8}$$

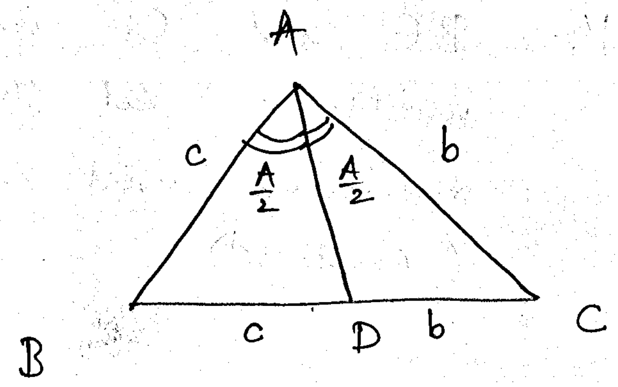


$$\left\{ \text{half the area of equilateral } \Delta\text{le} = \frac{\frac{1}{2} \times 1 \times \sin 60^\circ}{2} = \frac{\sqrt{3}}{8} \right\}$$

$$\text{Area of } \Delta ABC = 3 \times \text{Area of } \Delta BOC$$

$$= \frac{3\sqrt{3}}{8}$$

15.



AD bisects the angle A.

Thus, $\frac{AB}{AC} = \frac{BD}{CD} = \frac{c}{b}$

$\Rightarrow BD = a \cdot \frac{c}{c+b}$ and $CD = a \cdot \frac{b}{c+b}$

Area of $\Delta ABD = \frac{1}{2} \times AB \times AD \sin\left(\frac{A}{2}\right)$

$= AD \cdot \frac{c}{2} \sin\left(\frac{A}{2}\right)$

Similarly,

Area of $\Delta ACD = AD \cdot \frac{b}{2} \sin\left(\frac{A}{2}\right)$

Area of $\Delta ABC = \frac{1}{2} bc \sin A = AD \, \frac{b+c}{2} \sin\left(\frac{A}{2}\right)$

$\Rightarrow AD = \left(\frac{2bc}{b+c}\right) \cos\left(\frac{A}{2}\right)$

— end —