Week 21: Solutions to practice problems

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Solutions to properties of triangles (problem set 3)

http://bit.ly/trig2013

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1. $(b - a\cos C)\tan A = a\sin C$

Sol (i) ~~b tan A~~ We shall prove that

$$\tan A = \frac{a\sin C}{b - a\cos C} \quad \left[\text{assuming } b - a\cos C \neq 0\right]$$

$$\text{RHS} = \frac{\sin C}{\frac{b}{a} - \cos C}$$

$$= \frac{\frac{c}{2R}}{\frac{b}{a} - \frac{a^2+b^2-c^2}{2ab}}$$

$$= \frac{\frac{c}{2R}}{\frac{b^2-a^2+c^2}{2ab}} = \frac{\frac{a}{2R}}{\frac{b^2+c^2-a^2}{2bc}}$$

$$= \frac{\sin A}{\cos A} = \tan A$$

(Assuming $\cos A \neq 0$)

Note that if $b = a\cos C$

Then, $\cos C = \dfrac{b}{a}$

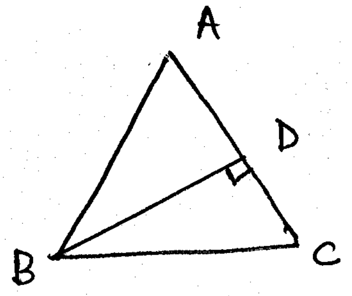
$$\Rightarrow \frac{b^2+a^2-c^2}{2ab} = \frac{b}{a}$$

$$\Rightarrow b^2+a^2-c^2 = 2b^2$$

$$\Rightarrow b^2+c^2 = a^2$$

$\Rightarrow$ Δle is right angled at vertex A.
Hence, $\tan A$ is undefined and $\cos A$ is 0.

(ii)



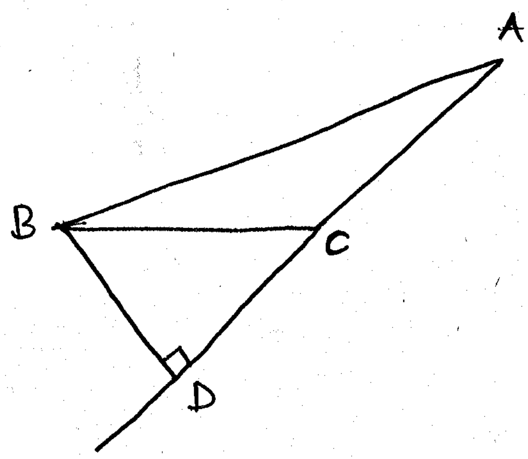
$(b - a\cos C)\tan A$

$= (AC - DC)\dfrac{BD}{AD}$

$= AD \times \dfrac{BD}{AD} = AD$

$= \cancel{c\sin A}$

$= a\sin C$



$(b - a\cos C)\tan A$

$= [b - (-CD)]\dfrac{BD}{AD}$

$= (b + CD)\dfrac{BD}{AD}$

$= AD \times \dfrac{BD}{AD} = BD = a\sin C$

Cases when $\underline{|B}$ and $\underline{|A}$ are obtuse are left are exercises.

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2. $b^2 \sin 2C + c^2 \sin 2B$

$$= 2\left[b^2 \sin C \cos C + c^2 \sin B \cos B \right]$$

$$= 2\left[b^2 \times \frac{c}{2R} \times \frac{b^2+a^2-c^2}{2ab} + c^2 \times \frac{b}{2R} \times \frac{a^2+c^2-b^2}{2ac} \right]$$

$$= \frac{2abc}{2R} = \frac{abc}{R} = \frac{bc\,(2R\sin A)}{R}$$

$$= 2bc \sin A$$

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3. $(b^2-a^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0$

Note that $\cot A = \dfrac{\cos A}{\sin A} = \dfrac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a}{2R}}$

$$= \frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a}{2\frac{abc}{4\Delta}}}$$

$$= \frac{b^2+c^2-a^2}{4\Delta}$$

On substituting cot A, cot B and c and adding them, we get the sum as zero.

4. $a^2+b^2+c^2 = 2(bc\cos A + ac\cos B + ab\cos C)$

$$\text{RHS} = \sum_{cyc} 2bc\cos A$$

$$= \sum_{cyc} (b^2+c^2-a^2) = a^2+b^2+c^2$$

5. Note that

$$\cot\left(\frac{B}{2}\right) = \frac{\cos(B/2)}{\sin(B/2)} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}$$

$$= \frac{s(s-b)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{s(s-b)}{\Delta}$$

Similarly, $\cot\left(\frac{C}{2}\right) = \frac{s(s-c)}{\Delta}$

$$3a = b+c$$

$$\Rightarrow 4a = a+b+c \Rightarrow 4a = 2s$$

or $\boxed{s = 2a}$

On the other hand,

$$\cot\frac{B}{2} \times \cot\frac{C}{2} = \frac{s^2(s-c)(s-b)}{\Delta^2}$$

$$= \frac{s^2(s-c)(s-b)}{s(s-a)(s-b)(s-c)}$$

$$= \frac{s}{s-a} = \frac{2a}{2a-a} = 2$$

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6. Note that,

$$\frac{(2a+1)^2 + (a^2-1)^2 - (a^2+a+1)^2}{2(2a+1)(a^2-1)} = -\frac{1}{2}$$

on simplification.

Hence, cos of some angle is $-1/2$. Hence, that angle is $\frac{2\pi}{3}$. As there can be atmost one obtuse angle in a triangle, the greatest angle is $\frac{2\pi}{3}$.

7. $$\cancel{\sum_{cyc} \sin^3 A \cos(B-C)}$$

$$\cancel{= \sum_{cyc} \sin^2 A \cdot \sin A}$$

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7. If $a \cos A = b \cos B$

$$2R \sin A \cos A = 2b \sin B \cdot \cos B$$

$$\Rightarrow 2R(\sin 2A - \sin 2B) = 0$$

$$\Rightarrow 4R \sin(A-B) \cos(A+B) = 0$$

$$\sin(A-B) = 0$$

$\Rightarrow A-B=0 \Rightarrow A=B$. The Δle is isoscles.

$$\cos(A+B) = 0$$

$\Rightarrow A+B = \pi/2 \Rightarrow C = \pi/2$. The Δle is right angled.

8. Suppose

$\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$, $\sin^2 \frac{C}{2}$ are in HP

$\Leftrightarrow$ $\frac{(s-b)(s-c)}{bc}$, $\frac{(s-a)(s-c)}{ac}$, $\frac{(s-b)(s-a)}{ab}$ in HP.

Multiplying ~~each~~ terms by $s-a$, $s-b$ and $s-c$ respectively and dividing by common factor,

$\Leftrightarrow$ $\frac{1}{bc(s-a)}$, $\frac{1}{ac(s-b)}$, $\frac{1}{ab(s-c)}$ are in HP.

Multiplying terms by a, b and c (on both numerator and denominator) and taking off the common factor,

$\Leftrightarrow$ $\frac{a}{s-a}$, $\frac{b}{s-b}$, $\frac{c}{s-c}$ are in ~~HP~~. HP.

Adding 1 to each

$\Leftrightarrow$ $\frac{s}{\cancel{s}a}$, $\frac{s}{\cancel{s}b}$, $\frac{s}{\cancel{s}c}$ are in HP

~~$\Leftrightarrow$ s-a, s-b, s-c are in AP~~

$\Leftrightarrow$ a, b, c are in AP.

9. $a \tan A + b \tan B = (a+b) \tan\left(\frac{A+B}{2}\right)$

$\Rightarrow \left[a\left(\tan A - \tan\left(\frac{A+B}{2}\right)\right)\right] = b\left[\tan\frac{A+B}{2} - \tan B\right]$

$$\Rightarrow \frac{a\left(\sin A \cos\frac{A+B}{2} - \cos A \sin\frac{A+B}{2}\right)}{\cos A \cos\frac{A+B}{2}} = \frac{b\left(\sin\frac{A+B}{2}\cos B - \cos\frac{A+B}{2}\sin B\right)}{\cos\frac{A+B}{2}\cos B}$$

Simplify,

$$\frac{a \sin\frac{A-B}{2}}{\cos A} = \frac{b\cos\frac{A-B}{2}}{\cos B} \qquad \left[\text{as } \cos\frac{A+B}{2} \neq 0\right]$$

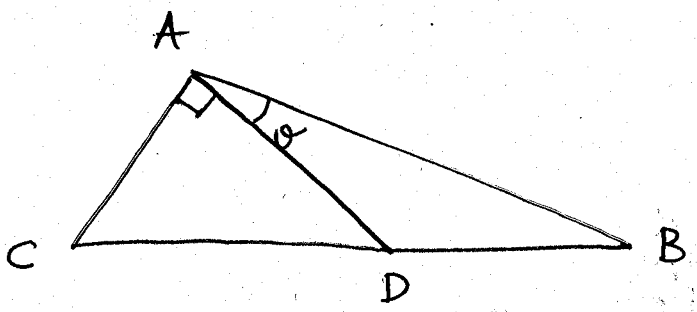
$\Rightarrow \sin\frac{A-B}{2}\left[2R \sin A \cos B - 2R \sin B \cos A\right] = 0$

$\Rightarrow 2R \sin\left(\frac{A-B}{2}\right) \sin(A-B) = 0$

$\Rightarrow A - B = 0$ or $\boxed{A = B}$

Hence, the tringle is isosceles.

10.



From Construction, $CD = DB$ and $\angle CAD = \pi/2$

Let $\angle DAB = \theta$.

From Δle ADB,

$$\frac{\sin\theta}{(a/2)} = \frac{\sin B}{AD} \quad \text{——} \quad (1)$$

From Δle ABC,

$$\frac{\sin(90+\theta)}{a} = \frac{\sin B}{b} \quad \text{——} \quad (2)$$

Dividing (1) by (2),

$$2\tan\theta = \frac{b}{AD}$$

$$\Rightarrow 2\tan(A-90^\circ) = \cot C$$

$$\Rightarrow -2\tan(90-A) = \cot C$$

$$\Rightarrow -2\tan C = \tan A$$

$$\Rightarrow \boxed{\tan A + 2\tan C = 0}$$