Week 19: Solutions

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1. $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = -\pi$

Answer: no solutions.

The ~~domain~~ range of $\cos^{-1}(x)$ is $[0, \pi]$

since $-\pi$ lies outside of that domain

and we are only adding '$-\pi$' will never occur

2. $\sin^{-1}(\sin x) = \pi - x$

c) $x \in [\pi/2, 3\pi/2]$

$$\boxed{\begin{array}{l}\text{Note that} \\ \sin^{-1}(\sin x) = x \text{ when } -\frac{\pi}{2} \le x < \frac{\pi}{2}\end{array}}$$

3. $\sum_{k=1}^{\infty} \tan^{-1}\left(\frac{2k}{k^4+k^2+2}\right)$ is

$\pi/2$ or $\pi/4$ or $\pi/3$ or none of the above

$$\frac{2k}{k^4+k^2+1+1}$$

$$\frac{\cancel{2k}}{(k^2+1)^2-k^2} = \frac{\cancel{2k}}{(k^2+1+k)(k^2+1-k)}$$

$$= \frac{(k^2+1+k)-(k^2+1-k)}{1+(k^2+1-k)(k^2+1+k)}$$

$$\Rightarrow \sum_{k=1}^{\infty} \tan^{-1}(k^2+1+k) - \tan^{-1}(k^2+1-k)$$

$$= \cancel{\tan^{-1}(3)} - \tan^{-1}(1)$$

$$+ \tan^{-1}(7) - \cancel{\tan^{-1}(3)}$$

$$\vdots$$

$$= \lim_{k\to\infty} \tan^{-1}(k^2+1+k) - \pi/4$$

$$= \pi/2 - \pi/2 = \pi/4$$

Ans Provided, x, y, z are all positive and $x^2+y^2+z^2=r^2$

3. $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{zx}{yr} + \tan^{-1}\frac{xy}{zr}$

$= \tan^{-1}\left(\dfrac{\frac{yz}{xr}+\frac{zx}{yr}}{1-\frac{z^2}{r^2}}\right) + \tan^{-1}\frac{xy}{zr}$

Test whether $\frac{yz}{xr} \times \frac{zx}{yr} < 1$

LHS $= \frac{z^2}{r^2} < 1$ is true.

$= \tan^{-1}\left(\frac{zr}{xy}\right) + \tan^{-1}\left(\frac{xy}{zr}\right)$ $\left[\text{from } \tan^{-1}(a) + \tan^{-1}\frac{1}{a} = \frac{\pi}{2}\right]$ where $a > 0$.

$= \frac{\pi}{2}$

(a)

Provided $-\sqrt{2} < x < 1/3$

4. $\tan^{-1}(x+1) + \tan^{-1}(x) + \tan^{-1}(x-1) = \tan^{-1}(3)$

$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}(3) - \tan^{-1}(x)$

$\tan^{-1}\left[\frac{x+1+x-1}{1-(x^2-1)}\right] = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$

test. $x^2-1 < 1$ $\Rightarrow x^2 < 2$ is true

$\tan^{-1}\left[\frac{2x}{2-x^2}\right] = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$

test, $3x < 1$ $x < 1/3$ is true

$\Rightarrow x^3 - 9x^2 - 4x + 6 = 0$

If $x = -1$ is a soln.

$-1 - 9 + 4 + 6 = 0$

$\therefore x = -1$ is a soln.

$$(x^3 - 9x^2 - 4x + 6) \div (x+1)$$

$$= x^2 - 10x + 6$$

$$x = \frac{10 \pm \sqrt{100 - 24}}{2}$$

$$= 5 \pm \frac{\sqrt{76}}{2}$$

$5 - \frac{\sqrt{76}}{2} > \frac{1}{3}$ and $5 + \frac{\sqrt{76}}{2} > \frac{1}{3}$

$\therefore$ (a) there is no solution other than $x = -1$

(c) option (d) is impossible

~~[(d) None of these]~~

Lecture 19: Problems on Inverse Trig Functions:

6) $\sum_{n\geq 1}^{\infty} \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$

equals to

a) $\frac{\pi}{2}$

b) $\frac{\pi}{4}$

c) $\frac{\pi}{3}$

d) None of these.

Soln: $\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}$; $\Rightarrow \frac{1}{\sqrt{n}}\sqrt{\frac{n}{n+1}} - \frac{1}{\sqrt{n+1}}\sqrt{\frac{n-1}{n}}$

$\Rightarrow \frac{1}{\sqrt{n}}\sqrt{\frac{n+1-1}{n+1}} - \frac{1}{\sqrt{n+1}}\sqrt{\frac{n-1}{n}} \Rightarrow \frac{1}{\sqrt{n}}\sqrt{1-\frac{1}{n+1}} - \frac{1}{\sqrt{n+1}}\sqrt{1-\frac{1}{n}}$

$\sin\theta = \frac{1}{\sqrt{n}}$, $\cos\theta = \sqrt{1-\frac{1}{n}}$ $\cos\alpha = \sqrt{1-\frac{1}{n+1}}$, $\sin\alpha = \sqrt{1-\left(1-\frac{1}{n+1}\right)}$

$\sin\alpha = \frac{1}{\sqrt{n+1}}$

$\therefore$ $\sin\theta\cos\alpha - \sin\alpha\cos\theta$ can be written as $\sin(\theta-\alpha)$

$\theta - \alpha = \sin^{-1}\left(\frac{1}{\sqrt{n}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$

From here it is a telescopic series;

$\sin^{-1}(1) - \cancel{\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)}$

$+ \cancel{\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\vdots$

We get $\sin^{-1}(1) - \cancel{\sin^{-1}(0)}$. $\sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$

Note : $\cancel{\sin^{-1}(0)}$ $\sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$ ~~was~~ is actually $\sin^{-1}\left(\frac{1}{\text{huge no.}}\right)$, tending to 0.

$\therefore$ Answer a) $\frac{\pi}{2}$.

7) $\sin^{-1}(x) + \sin^{-1}(y) + \sin^{-1}(z) = \pi$

If $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = k(xyz)$, then k is

a) 1
b) 2
c) 3
d) None of these.

Soln: let, $\sin A = x$, $\sin B = y$, $\sin C = z$.

$\therefore \sin A\cos A + \sin B\cos B + \sin C\cos C = k\sin A\sin B\sin C$.

$\sin 2A + \sin 2B + \sin 2C = 2k\sin ABC$.

$L.H.S = 2\sin(A+B)\cos(A-B) + \sin 2(\pi - (A+B))$

$-\sin 2(A+B)$

$2\sin(A+B)\cos(A+B)$

$\Rightarrow 2\sin(A+B)\left(\cos(A-B) - \cos(A+B)\right)$

$\Rightarrow 2\sin C \times 2\sin A\sin B = 4\sin A\sin B\sin C$.

$\therefore 4 = 2k$

$k = \underline{\underline{2}}$.

Answer b) $\underline{\underline{2}}$.

8) The interval where $\sin^{-1}(x) < \cos^{-1}(x)$ is $\left(0 < x < \frac{1}{\sqrt{2}}\right.$

a) $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

b) $0 < x < \frac{+1}{\sqrt{2}}$

c) $\frac{1}{\sqrt{2}} < x < 1$

d) None of these

Soln: ~~By substituting values; we get d)~~
(b)

9) $\left(\tan^{-1}(x)\right)^2 + \left(\cot^{-1}(x)\right)^2 = \frac{5\pi^2}{8}$

a) has exactly one soln.

b) has exactly two solns.

c) has no solns.

d) None of these.

Soln: $\left(\tan^{-1}x + \cot^{-1}x\right)^2 - 2\tan^{-1}x\cot^{-1}x = \frac{5\pi^2}{8}$

$\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\Rightarrow \frac{\pi^2}{4} - 2\tan^{-1}x\left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{5\pi^2}{8}$

let $y = \tan^{-1} x$
we get the quadratic eqn:

$2y^2 - \pi y - \frac{3\pi^2}{8} \Rightarrow$ roots are $\frac{3\pi}{4}$ & $-\pi/4$.

$\because \frac{3\pi}{4}$ is out of tan's principle domain.

$\therefore$ Only 1 soln.

Answer is a).

10) If $[\sin^{-1}(\tan^{-1} x)] = 1$

Then,

a) $x \in [0, \tan(\pi - 1)]$

b) $x \in [-\infty, \sin^{-1}(1)]$

c) $x \in [\tan(\sin(1)), \infty]$

d) None of these.

Soln :

$$1 \leq \sin^{-1}(\tan^{-1}(x)) \leq \frac{\pi}{2}$$

$$\Rightarrow \sin 1 \leq \tan^{-1} x \leq 1$$

$$\Rightarrow \tan(\sin 1) \leq x \leq \tan 1$$

Answer d) None of these.

11) If $p > q > 0$ and $pr < -1 < qr$, then

$$S = \tan^{-1}\left(\frac{p-q}{1+pq}\right) + \tan^{-1}\left(\frac{q-r}{1+qr}\right) + \tan^{-1}\left(\frac{r-p}{1+rp}\right)$$

a) 0
b) π
c) $\frac{\pi}{2}$
d) None of these

~~solution~~ : b.

Solution: From the second inequality,

$(p-q)r < 0$

Since $p-q > 0$ from the first inequality,

we get $\boxed{r < 0}$

$$\tan^{-1}\left(\frac{p-q}{1+pq}\right) = \tan^{-1}(p) - \tan^{-1}(q) \quad \left[\text{as } \frac{p-q}{1+pq} \text{ is +ve}\right]$$

$$\tan^{-1}\left[\frac{q-r}{1+qr}\right] = \tan^{-1}(q) - \tan^{-1}(r) \quad \left[\begin{array}{l}\text{as } q-r \text{ is +ve} \\ 1+qr \text{ is +ve}\end{array}\right]$$

$$\tan^{-1}\left[\frac{r-p}{1+rp}\right] = \pi + \tan^{-1}(r) - \tan^{-1} p \quad \left[\begin{array}{l}\text{as } r-p \text{ is -ve} \\ \text{and} \\ 1+rp \text{ is -ve} \\ \text{further } rp < -1\end{array}\right]$$

Hence, $\boxed{S = \pi}$

8. $\sin^{-1}(x) < \cos^{-1}(x)$ ——— (1)

Note that $-1 \leq x \leq 1$.

Let $\sin^{-1} x = \theta$ and $\cos^{-1} x = \varphi$

(1) $\Rightarrow$ $\theta < \varphi$ ——— (2)

Applying sin on both sides of (1), we get

$$\sin(\sin^{-1} x) < \sin(\cos^{-1} x)$$

$$x < \sin(\sin^{-1}\sqrt{1-x^2})$$

$$\Rightarrow \quad x < \sqrt{1-x^2} \quad \text{——— (3)}$$

Note that (3) is true when $-1 \leq x \leq 0$ —(4)

When $x > 0$ ie., $0 < x \leq 1$,

(3) $\Rightarrow$ $x^2 < 1 - x^2$

$\Rightarrow$ $2x^2 < 1$

$\Rightarrow$ $x^2 < \frac{1}{2}$

$\Rightarrow$ $x < \frac{1}{\sqrt{2}}$

Hence, the required interval is $[-1, 0] \cup \left(0, \frac{1}{\sqrt{2}}\right)$

12. We have,

$$2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z \quad \text{--- (1)}$$

and $\quad 2y = x + z \quad$ --- (2)

From (1),

$$\tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

(as $xz < 1$, RHS is valid, we need not add a π)

(as $y^2 < 1$, LHS is valid)

$$\Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-xz} \quad \text{--- (3)}$$

From (3) and (2)

$1 - y^2 = 1 - xz$

$\Rightarrow y^2 = xz$

Hence, x, y, z are in GP as well as AP --- (4)

(4) is possible only when $x = y = z$

Hence, option (a) and (c) are correct.

— end —