Lecture 16: Summation of finite trigonometric series

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An elementary 'telescoping sum'

Example: Find the sum(finite) of

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}$$

Solution.

$$\begin{aligned} LHS &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= 1-\frac{1}{n+1} \end{aligned}$$

Corollary: What about the infinite sum

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots$$

Answer. As $n\to\infty$, $\left(1-\frac{1}{n+1}\right)\to 1$

Result:

$$\underbrace{\sin a + \sin(a+b) + \sin(a+2b) + \cdots + \sin(a+(n-1)b)}_{n \text{ terms}}$$

Solution.

$$= \frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{b}{2}\right)} \left(\sin a + \sin(a+b) + \sin(a+2b) + \cdots + \sin(a+(n-1)b)\right)$$

$$= \frac{1}{\sin\left(\frac{b}{2}\right)} \left(\sin a \sin\left(\frac{b}{2}\right) + \sin(a+b)\sin\left(\frac{b}{2}\right) + \cdots + \sin(a+(n-1)b)\sin\left(\frac{b}{2}\right)\right)$$

$$= \frac{1}{2\sin\left(\frac{b}{2}\right)} \left(\cos\left(a-\frac{b}{2}\right) - \cos\left(a+\frac{b}{2}\right) + \cos\left(a+\frac{b}{2}\right) - \cos\left(a-\frac{3b}{2}\right) + \dots\right.$$

$$\left.\cdots + \cos\left(a+\frac{(2n-3)b}{2}\right) - \cos\left(a-\frac{(2n-1)b}{2}\right)\right)$$

$$= \frac{1}{2\sin\left(\frac{b}{2}\right)} \left(\cos\left(a-\frac{b}{2}\right) - \cos\left(a-\frac{(2n-1)b}{2}\right)\right)$$

$$= \frac{1}{\sin\left(\frac{b}{2}\right)} \left(\sin\left(a+\frac{n-1}{2}b\right)\sin\left(\frac{nb}{2}\right)\right)$$

We have,

$$\underbrace{\sin a + \sin(a+b) + \sin(a+2b) + \cdots + \sin(a+(n-1)b)}_{n \text{ terms}}$$

$$= \frac{\sin\left(\frac{nb}{2}\right)}{\sin\left(\frac{b}{2}\right)} \sin\left(a + \frac{(n-1)b}{2}\right)$$

$$= \frac{\sin\left(\frac{n \text{ diff}}{2}\right)}{\sin\left(\frac{\text{diff}}{2}\right)} \sin\left(\frac{\text{first + last}}{2}\right)$$

Similarly, we can prove

$$\underbrace{\cos a + \cos(a+b) + \cos(a+2b) + \cdots + \cos(a+(n-1)b)}_{n \text{ terms}}$$

$$= \frac{\sin\left(\frac{nb}{2}\right)}{\sin\left(\frac{b}{2}\right)} \cos\left(a + \frac{(n-1)b}{2}\right)$$

$$= \frac{\sin\left(\frac{n \text{ diff}}{2}\right)}{\sin\left(\frac{\text{diff}}{2}\right)} \cos\left(\frac{\text{first + last}}{2}\right)$$

Example: Find

$$S = \cos a + \cos 3a + \cdots + \cos(2n-1)a$$

Solution.

$$S = \frac{\sin na}{\sin a}\cos\left(\frac{a+(2n-1)a}{2}\right)$$
$$= \frac{\sin na \cos na}{\sin a} = \frac{\sin 2na}{2\sin a}$$

Example:

$$\underbrace{\sin a + \cos(a+b) - \sin(a+2b) - \cos(a+3b) + \sin(a+4b) + \cos(a+5b) - \dots}_{n \text{ terms}}$$

Solution.

$$= \sin a + \sin(a + b + \frac{\pi}{2}) + \sin(a+2b+\pi) + \sin(a+3b+\frac{3\pi}{2}) + \dots$$
$$= \frac{\sin\left(\frac{n(2b+\pi)}{4}\right)}{\sin\left(\frac{(2b+\pi)}{4}\right)} \sin\left(a + \frac{(n-1)(2b+\pi)}{4}\right)$$

Problem. Find the sum upto n terms:

$$\sin x - \sin 2x + \sin 3x - \sin 4x + \ldots$$

Solution. We may rewrite as,

$$\sin x + \sin(2x + \pi) + \sin(3x + 2\pi) + \sin(4x + 3\pi) + \cdots + \sin(nx + (n-1)\pi)$$
$$= \frac{\sin\left(\frac{n(x+\pi)}{2}\right)}{\sin\left(\frac{(x+\pi)}{2}\right)} \sin\left(\frac{x + nx + (n-1)\pi}{2}\right)$$
$$= \frac{\sin\left(\frac{n(x+\pi)}{2}\right)}{\cos\left(\frac{x}{2}\right)} \sin\left(\frac{(n+1)x + (n-1)\pi}{2}\right)$$

Problem. If

$$2\sin 2a\left(\sin a\cos 3a+\sin 3a\cos 5a+\cdots+\sin(2n-1)a\cos(2n+1)a\right)$$
$$=\sin kna\sin ka(n+1)-n\sin^2 ka$$

Find k

Solution.

$$\begin{aligned}
&\sin a\cos 3a+\sin 3a\cos 5a+\cdots+\sin(2n-1)a\cos(2n+1)a\\
&=\frac{1}{2}\left(\sin 4a-\sin 2a+\sin 8a-\sin 2a+\cdots+\sin 4na-\sin 2a\right)\\
&=\frac{1}{2}\left(\sin 4a+\sin 8a+\cdots+\sin 4na-n\sin 2a\right)\\
&=\frac{1}{2}\frac{\sin 2na}{\sin 2a}\sin(2a+2na)-n\sin 2a
\end{aligned}$$

Hence, $k=2$

Problem.

$$\sin^2 x \sin 2x + \frac{1}{2}\sin^2 2x \sin 4x + \frac{1}{4}\sin^2 4x \sin 8x + \ldots (\text{up to n terms})$$

Solution.

$$= \left(\frac{1-\cos 2x}{2}\right)\sin 2x + \frac{1}{2}\left(\frac{1-\cos 4x}{2}\right)\sin 4x + \ldots$$

$$= \frac{\sin 2x}{2} - \frac{\sin 4x}{4} + \frac{\sin 4x}{4} - \frac{\sin 8x}{8} + \ldots$$

$$= \frac{\sin 2x}{2} - \frac{\sin 2^{n+1}x}{2^{n+1}}$$

What happens when $n \to \infty$?

Problem. Prove that

$$\tan x = \cot x - 2\cot 2x$$

and then find the sum

$$\sum_{k=1}^{n} 2^k \tan(2^k x)$$

What happens when $n \to \infty$?

Solution.

$$\sum_{k=1}^{n} 2^k \tan(2^k x) = \sum_{k=1}^{n} \cot 2^k x - 2\cot 2^{k+1} x$$
$$= 2\cot 2x - 2^{n+1} \cot 2^{n+1} x$$

We cannot say anything as $n \to \infty$

Problem. Expand

$$\sum_{\text{n terms}} \sec a \sec 2a = \sec a \sec 2a + \sec 2a \sec 3a + \dots$$

where $a \neq n\pi$

Solution.

$$\sum_{\text{n terms}} \sec a \sec 2a = \sum \frac{1}{\cos a \cos 2a}$$
$$= \sum \frac{\sin a \csc a}{\cos a \cos 2a}$$
$$= \sum \csc a \left(\frac{\sin(2a - a)}{\cos a \cos 2a} \right)$$
$$= \sum \csc a (\tan 2a - \tan a) \quad = \csc a (\tan(n+1)a - \tan a)$$

I am going to skip these problems

1. $\csc x \csc 3x + \csc 3x \csc 5x + \ldots$
2. $\sin a \sec 3a + \sin 3a \sec 9a + \ldots$
3. $\cos^3 a + \cos^3 (a - 2\pi/n) + \cos^3 (a - 4\pi/n) + \ldots$
4. $$\frac{1}{\cos a + \cos 3a} + \frac{1}{\cos a + \cos 5a} + \frac{1}{\cos a + \cos 7a} + \ldots$$
5. Find the maximum and minimum values of
   - $$\frac{\tan^2 x + \cot^2 x + 1}{\tan^2 x + \cot^2 x - 1}$$
   - $\sin A + \sin B + \sin C$ where A, B, C are the angles of a triangle.
   - $a \cos(b + x) + c \sin(d + x)$ where a, b, c, d are real numbers.
   - $a^2 \tan^2 x + b^2 \cot^2 x$