Lecture 13: Problems on functions

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Problem.

The minimum value of the function $f : (0, \infty) \to \mathbb{R}$ such that $f(x) = x + \frac{1}{x}$ is
(a) 1
(b) 2
(c) There is no minimum value
(d) None of these

Answer. (b)

Solution. Let $y = x + \frac{1}{x}$. Since, $x > 0$, we have $y > 0$. Writing the equation as a quadradic,

$$x^2 - yx + 1 = 0$$

Note that x has a real solution. Hence, the discriminant is non-negative. We have,

$$y^2 - 4 \geq 0$$

$$\Rightarrow y \geq 2$$

Hence, the minimum value of $f(x)$ is 2.

Problem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x) = \begin{cases} 0 & \text{,if x is irrational} \\ 1 & \text{,if x is rational} \end{cases}$$

Then, the function f is

(a) periodic with a period

(b) not periodic

(c) periodic but without a period

(d) period depends on x

Answer. (c)

Solution. Note that $f(r+1) = f(r') = 1$ where r is a rational number. Similarly, $f(i+1) = f(i') = 0$ where i is an irrational number. Together, $f(x+1) = f(x) \quad \forall x \in \mathbb{R}$. Hence, f is periodic.

But note that we can replace by any rational number like $\{1/2, 1/3, 1/4, \ldots\}$. Since, we do not have the "smallest positive rational number", we cannot assign a period for the function.

Problem.

Let $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x) = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

The image (range) of the function is,
(a) $[1/7, 7]$
(b) $[-1/7, 7]$
(c) $[-7, 7]$
(d) $(-\infty, 1/7) \cup (7, \infty)$

Answer. (a)

Solution. Let $y = f(x)$

$$y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

$$(y-1)x^2 + 3(y+1)x + 4(y-1) = 0$$

$$9(y+1)^2 - 16(y-1)^2 \geq 0 \quad \text{the Discriminant is non-negative}$$

$$-7y^2 + 50y - 7 \geq 0$$

$$\Rightarrow y \in \left[\frac{1}{7}, 7\right]$$

Problem.

Suppose $f : \mathbb{R} \to \mathbb{R}$ such that $2f(x) + 3f(-x) = 1 + x \quad, \forall x \in \mathbb{R}$. Then, f is

(a) there does not exist such a f

(b) f is a linear function of x

(c) f is not bijective

(d) None of these

Answer. (b)

Solution. Put $-x$ in place of x,

$$2f(x) + 3f(-x) = 1 + x \quad \text{original equation}$$
$$2f(-x) + 3f(x) = 1 - x \quad \text{after substituting } -x \text{ in place of } x$$

Solving the simultaneous equations, $f(x) = \frac{1}{5} - x$

Problem.

$f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3 - x$ is

(a) ~~inj~~, sur

(b) inj, ~~sur~~

(c) ~~inj~~,~~sur~~

(d) inj, sur

Answer. (a)

Solution. Note that $0 \to 0$ and $1 \to 0$. Hence the function is not injective.

By intermediate value theorem, any odd degree polynomial function in x is surjective.

(more than one option can be correct)

Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Then, f is

(a) f is surjective

(b) f is injective

(c) f cannot miss exactly one point in $\mathbb{R}$

(d) f cannot miss exactly two points in $\mathbb{R}$

Answer. (b),(c),(d)

Solution. Since, f is strictly increasing, the same function value never repeats. Hence, the function is injective.

Consider the strictly increasing function $f : \mathbb{R} \to \mathbb{R}, f(x) = 10^x$. We know that $10^x > 0$. Hence, f need not be surjective.

f have only jump discontinuities, if it is discontinuous. Then, we miss an interval of values. We cannot miss only a finite set of values.

Problem. Let a, b, c be fixed positive numbers such that $f : \mathbb{N} \to \mathbb{R}$

$$f(n) = \frac{na}{b + nc}$$

Then,

(a) f is increasing

(b) f is decreasing

(c) f increases first and then decreases

(d) None of these

Answer. (a)

Solution. Since, n is never zero, we rewrite the function by dividing numerator and denominator by n as

$$f(n) = \frac{a}{\frac{b}{n} + c}$$

Note that $\frac{b}{n}$ decreases as n increases. Hence, f increases.

Problem. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(a) + f(b) = f(a)f(b)$$

Then,

(a) The image(range) cannot have more than one element.

(b) The image(range) cannot have more than two elements.

(c) The image set can have infinitely many elements.

(d) We cannot say anything about the image set.

Answer. (b)

Solution. We shall understand more about the function by substituting values. Let $x = a = b$. Then,

$$2f(x) = (f(x))^2$$

This equation as two solutions.

$$f(x) = 0, f(x) = 2$$

Hence, image of f cannot have more than two elements.

Problem. The symmetric difference of two sets is defined as $A * B := (A \setminus B) \cup (B \setminus A)$.

Let R be a relation on the set $2^{\mathbb{N}} :=$ Set of all subsets of $\mathbb{N}$. We have ${}_A R_B$ if $A * B \neq \emptyset$.

(a) ~~reflexive~~, symmetric ,transitive

(b) reflexive, symmetric ,~~transitive~~

(c) ~~reflexive~~, ~~symmetric~~ ,~~transitive~~

(d) ~~reflexive~~, symmetric ,~~transitive~~

Answer. (d)

Solution. Note that $\emptyset * \emptyset = \emptyset$. Hence, the relation is not reflexive.

$A * B \neq \emptyset \Rightarrow B * A \neq \emptyset$. Hence, the relation is symmetric.

Utilising, the fact that relation is not reflexive, let $A = \{1\}, B = \{2\}$, we have ${}_A R_B$ and ${}_B R_A$, but A is not related to itself.

Problem. Pick the function that does not belong to the category. (observe their period)

$$1. \frac{1+\sin x}{\cos x(1+\csc x)} \quad 2. \mid \sin \mid x \mid\mid \quad 3. \frac{1}{1-\cos 2x} \quad 4. x+\cos x$$

(a) 1

(b) 2

(c) 3

(d) 4

Answer. (d)

Solution. Note that the first function is $\tan x$. First three functions have period π. The last function is not periodic as it has a linear term x (increasing function).

Problem. Lets define $| f | := | f(x) |$ and

$$max(f, g)(x) := max(f(x), g(x))$$

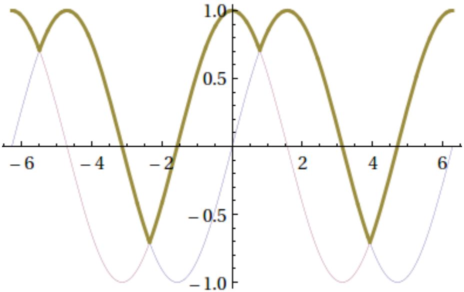
and

$$min(f, g)(x) := min(f(x), g(x))$$

Then, $max(f, g)(x)$ in terms of $| f |$ and $| g |$ is ......

Answer.

$$max(f, g)(x) = \frac{(f + g) + | f - g |}{2}$$



Problem.

Let $f : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y)$$

Then,

(a) f is a non-linear function

(b) $f(x) \equiv 0$

(c) $f(x) = cx$ for some $c \in \mathbb{R}$

(d) None of these

Answer. (c)

Solution. Put $x = 0$, we have $f(0) = 0$. Note that when n is a positive natural number, $f(n) = f(\underbrace{1 + \cdots + 1}_{\text{n times}}) = nf(1) = nc$ where $c = f(1)$, also

$f(n + (-n)) = f(n) + f(-n) = 0$. Hence, $f(-n) = (-n)c$.

Extending the argument to rational numbers,

$$mc = f(m) = f\left(\underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{n \text{ times}}\right) = nf\left(\frac{m}{n}\right)$$

$$\Rightarrow f\left(\frac{m}{n}\right) = c\left(\frac{m}{n}\right)$$

Problem. Let f be a polynomial in x of degree greater than 0. Which of the statements is true?

(a) $f(a) = 0 \Leftrightarrow (x - a)$ is a factor of $f(x)$

(b) $f(a) = 0 \Rightarrow (x - a)$ is a factor of $f(x)$

(c) $f(a) = 0 \Leftarrow (x - a)$ is a factor of $f(x)$

(d) $f(a) = 0 \not\Leftrightarrow (x - a)$ is a factor of $f(x)$

Answer. (a)

Let $x - a$ be a factor of $f(x)$. Then, $f(x) = g(x)(x - a)$. Hence, $f(a) = 0$. This proves $\Leftarrow$ part.

In general, $f(x) = h(x)(x - a) + r(x)$ where $r(x)$ is the remainder polynomial whose degree is less than the degree of $x - a$. Hence, $r(x)$ is a constant. Let $r(x) = r$ for some real number r. Now, $f(x) = 0 \Rightarrow r = 0$ proving the $\Rightarrow$ part.

Problem. Let $I(x) = x \quad \forall\, x$. If $fog(x) = gof(x) = g(x) \quad \forall\, g(x)$. Then,

(a) $f = I$

(b) $f = I$ is just a possibility, f can be something else too.

(c) There is no such f

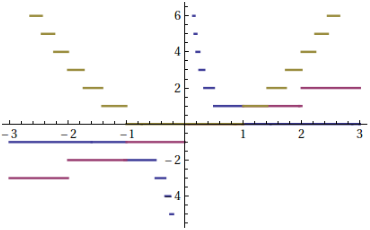
(d) None of these

Answer. (a)

Solution. Note that $Iog = goI = g \quad \forall g$. Suppose there are two different functions I and I' such that $Iog = goI = g \forall g$ and $I'og = goI' = g \forall g$.

Then, $IoI' = I'$ and $IoI' = I$. Hence, $I = I'$

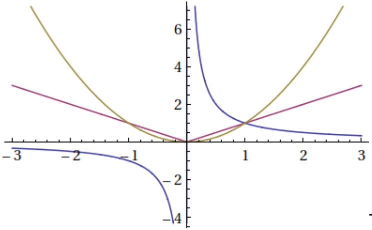
Problem. Which of these plots is the correct depiction of $\lfloor \frac{1}{x} \rfloor$?



(a) blue (b) yellow (c) violet (d) None of these

Answer. (a)

Solution. Observe that happens without taking $\lfloor \rfloor$ function.



The functions are $\frac{1}{x}, x, x^2$ respectively.

Problem. A function F is said to be increasing if $F(x) \geq F(y)$ whenever $x > y$ and strictly increasing if $F(x) > F(y)$ whenever $x > y$ (we can similarly define strictly decreasing).

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous (we can draw the graph of the function without lifting the pen) and f is non-negative when x is positive and f is non-positive when x is negative. Then, which of these is not possible?

(a) f is increasing

(b) f is strictly increasing

(c) f is strictly decreasing

(d) f is decreasing

Answer. (c)

Solution. Note that $f(0) = 0$ as the function is continuous. Write a picture to notice that (c) is not possible.

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