(Week 12) Class Handout: Relations and max/min of some trigonometric functions

http://bit.ly/trig2013

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The points here are sketchy. You may have to fill in a lot of details.

1 Relations

1. The cartesian product of two sets A and B is the set of all ordered pairs with the first element from A and second element from B.

   $$A \times B = \{(a, b) : a \in A, b \in B\}$$

   Example: Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Then, the cartesian product is $\{\{a, 1\}, \{a, 2\}, \{a, 3\}, \{b, 1\}, \{b, 2\}, \{b, 3\}, \{c, 1\}, \{c, 2\}, \{c, 3\}\}$

2. A relation from set A to set B is a subset of the cartesian product $A \times B$. If $a \to b$, we say ${}_aR_b$

3. A relation is a generalization of function, in sense that we do not have the restrictions:

   - Every element in the domain is mapped to some element of the co-domain.
   - Every element in the domain maps to exactly one element of the co-domain.

4. Properties of a relation $R : S \to S$:

   - relation is reflexive if ${}_aR_a \, \forall a \in S$
   - relation is symmetric if ${}_aR_b \Rightarrow {}_bR_a$
   - relation is transitive if ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c$

5. A relation which is reflexive, symmetric and transitive is called 'equivalence' relation.

6. Examples: Test whether the relations are reflexive, symmetric and transitive

   - Let R be a relation on $\mathbb{R}$ such that ${}_aR_b$ if $ab \geq 0$.

     (a) Note that $a^2 \geq 0 \quad \forall a \in \mathbb{R}$. Hence, R is reflexive.

     (b) If $ab \geq 0 \Rightarrow ba = ab \geq 0$. Hence, R is symmetric.

     (c) ${}_1R_0$ and ${}_0R_{-1}$, but ${}_1R_{-1}$ is not true. Hence, not transitive.

     What happens if we replace $ab \geq 0$ by $ab < 0$

   - Let R be a relation on $\mathbb{R}$ such that ${}_aR_b$ if $| a - b | > 1$

     (a) Note that $| a - a | < 1$. Hence, the relation is not reflexive.

     (b) If $| a - b | > 1$, we have $| b - a | = | a - b | > 1$. Hence, the relation is symmetric.

     (c) Note that ${}_1R_2$ and ${}_2R_{0.5}$, but ${}_1R_{0.5}$ is not true. Hence, the relation is not transitive.

     What happens if we replace $| a - b | > 1$ by $| a - b | \leq 1$

   - Let R be a relation on $\mathbb{N}$ such that ${}_aR_b$ if $a + 5b$ is even.

     (a) $a + 5a = 6a$ is always even. Hence, the relation is reflexive.

(b) Let $a+5b$ is even.

Case 1: Let a be even. Then, $5b$ should be even too. Hence, b is even. Further, $5a+b$ is even.

Case 2: Let a be odd. Then, $5b$ should be odd too. Hence, b is odd. Further, $5a+b$ is even.

Hence, the relation is symmetric. (the essence the argument was in recognising the fact that a and b have same parity)

(c) Let $a+5b$ and $b+5c$ be even. Note that all of them (a,b,c) are of same parity. Hence, its transitive.

7. Let R be an equivalence relation on set S. We define 'the equivalence class of an element $a \in S$' denoted by $[a]$ as the set of all elements related to a. Note that $[a] \neq \emptyset$ as $a \in [a]$.

8. Some facts without a proof: Let R be an equivalence relation on set S. If ${}_aR_b$, then $[a] \cap [b] = [a] = [b]$. If ${}_aR_b$ is false, then $[a] \cap [b] = \emptyset$. The set all equivalence classes partition the set S.

9. Exercises: Test whether the relations are reflexive, symmetric and transitive

   - Let R be a relation on $\mathbb{N}$ such that ${}_aR_b$ if $hcf(a,b) = 1$
   - Let R be a relation on $\mathbb{N}$ such that ${}_aR_b$ if $lcm(a,b) > max(a,b)$
   - Let R be a relation on set of all finite subsets of $\mathbb{N}$ except the emptyset such that ${}_AR_B$ if $n(A \cap B) \geq 1$ (where $n(X)$ stands for the number elements in set X)

2 Finding maximum and minimum of some trigonometric functions

1. Let $a, b \in \mathbb{R}$. Find the maximum and minimum value of $f(x) = a\sin x + b\cos x$

$$
\begin{aligned}
a\sin x + b\cos x &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)\\
&= \sqrt{a^2+b^2}\,(\cos\theta\sin x + \sin\theta\cos x)\\
&= \sqrt{a^2+b^2}\,\sin(\theta + x)
\end{aligned}
$$

where $\tan\theta = \frac{b}{a}$

Hence, the maximum and minimum values of the function are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

2. Examples: Find the max/min of these functions.

   - $f(x) = 4\sin x\cos x + 6(1 - 2\cos^2 x)$

     Observe that $f(x) = 2\sin 2x - 6\cos 2x$. Hence, $\sqrt{a^2+b^2} = \sqrt{4+36} = 2\sqrt{10}$. The max and min are $2\sqrt{10}$ and $-2\sqrt{10}$ respectively.

   - $f(x) = \cos 2x + \cos x$

     Note that both $\cos x$ and $\cos 2x$ equal to 1, when $x = 0$. Hence, the max value is 2.

     To find the minimum:

$$
\begin{aligned}
\cos 2x + \cos x = 2\cos^2 x - 1 + \cos x \quad &\text{(here is an invitation to complete the square)}\\
&= (\sqrt{2}\cos x)^2 + 2\times\frac{1}{2\sqrt{2}}\times\sqrt{2}\cos x - 1 + \left(\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{1}{2\sqrt{2}}\right)^2\\
&= \left(\sqrt{2}\cos x + \frac{1}{2\sqrt{2}}\right)^2 - \left(1 + \frac{1}{8}\right)\\
&= 2\left(\cos x + \frac{1}{4}\right)^2 - \frac{9}{8}
\end{aligned}
$$

     Note that term with the square acquires its minimum value, 0 when $\cos x = -\frac{1}{4}$. Hence the minimum value of the function is $-\frac{9}{8}$.

     We can find the maximum value by maximizing the square term, by putting $\cos x = 1$.