Solutions for problems from lecture-4

1, 2, 3 are usual derivations, find them in any textbook.

4. $\sin 15^\circ$

<u>Sol</u> $\sin 15^\circ = \sin(45-30)^\circ$

$= \sin 45 \cos 30 - \cos 45 \sin 30$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

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5. $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B$

LHS $= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= (\cos A \cos B)^2 - (\sin A \sin B)^2$

$= \cos^2 A \cos^2 B - (1-\cos^2 A)(1-\cos^2 B)$

$= \cos^2 A \cos^2 B - (1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B)$

$= \cos^2 A - 1 + \cos^2 B$

$= \cos^2 A - (1-\cos^2 B)$

$= \cos^2 A - \sin^2 B$

6. If

$$(1+\tan 1^\circ)(1+\tan 2^\circ) \cdots (1+\tan 45^\circ) = 2^n$$

Find n.

Sol Consider, $(1 \le x \le 44)$

$$1+\tan x^\circ = 1+\tan\left[45-(45-x)\right]$$

$$= 1 + \frac{\tan 45 - \tan(45-x)}{1+\tan 45 \tan(45-x)}$$

$$= 1 + \frac{1-\tan(45-x)}{1+\tan(45-x)}$$

$$= \frac{1+\tan(45-x)+1-\tan(45-x)}{1+\tan(45-x)}$$

$$= \frac{2}{1+\tan(45-x)}$$

$$(1+\tan 1^\circ)(1+\tan 2^\circ) \cdots (1+\tan 22^\circ)(1+\tan 23^\circ) \cdots (1+\tan 44^\circ)(1+\tan 45^\circ)$$

$$= \frac{2}{1+\tan 44^\circ} \times \frac{2}{1+\tan 43} \times \cdots \times \frac{2}{1+\tan 23^\circ} \times (1+\tan 23) \cdots \times (1+\tan 44) \times 2$$

$$= 2^{22} \times 2 = 2^{23}$$

Hence, $n = \cancel{23}\ 23$

7. Prove that

$$1 - \cot 23^\circ = \frac{2}{1 - \cot 22^\circ}$$

$$\Leftrightarrow \quad \frac{\tan 23 - 1}{\tan 23} = \frac{2 \tan 22}{\tan 22 - 1} \quad \left(\begin{array}{l}\text{on converting} \\ \text{cot to tan}\end{array}\right)$$

$$\Leftrightarrow \quad \tan 23 \tan 22 - \tan 22 - \tan 23 + 1 = 2 \tan 22 \tan 23 \quad \left(\begin{array}{c}\text{on cross} \\ \text{multiplication}\end{array}\right)$$

$$\Leftrightarrow \quad \tan 22 + \tan 23 = 1 - \tan 22 \tan 23$$

$$\Leftrightarrow \quad \frac{\tan 22 + \tan 23}{1 - \tan 22 \tan 23} = 1$$

$$\Leftrightarrow \quad \tan (22 + 23) = \tan 45 = 1$$

Since, the last statement is true, our claim is proved.

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8. $\sin x + \cos x$ (express as a single angle)

$$= \left[\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}}\right] \sqrt{2}$$

$$= \sqrt{2} \left[\sin x \cos 45^\circ + \cos x \sin 45^\circ\right]$$

$= \sqrt{2} \sin(x+45)$

Since, max and min values of sin is 1 and -1,

$-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}$.

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9. $\tan 20 + \tan 40 + \sqrt{3} \tan 20 \tan 40 = \sqrt{3}$

$\Leftrightarrow \tan 20 + \tan 40 = \sqrt{3}(1 - \tan 20 \tan 40)$

$\Leftrightarrow \dfrac{\tan 20 + \tan 40}{1 - \tan 20 \tan 40} = \sqrt{3}$

$\Leftrightarrow \tan(20+40) = \tan 60 = \sqrt{3}$

Since, the last statement is true, the claim is true.

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10. $\tan 3A - \tan 2A - \tan A = \tan 3A \cdot \tan 2A \cdot \tan A$

$\Leftrightarrow -\tan 2A - \tan A = \tan 3A \cdot \tan 2A \cdot \tan A - \tan 3A$

multiplying by -1 throughout and taking $\tan 3A$ common on RHS,

$\Leftrightarrow \tan 2A + \tan A = \tan 3A(1 - \tan 2A \tan A)$

Assuming $1 - \tan 2A \tan A \neq 0$,

$\Leftrightarrow \dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \tan(2A + A) = \tan 3A$.

Since, the last statement is correct, the claim is true.

11. Let A, B, C be the angles of a Δle.

Then, (assuming none of A, B, C are $\pi/2$)

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Sol

$$\Leftrightarrow \tan A + \tan B = \tan A \tan B \tan C - \tan C$$

$$= -\tan C\,(1 - \tan A \tan B)$$

$$\Leftrightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A+B)$$

$$= \tan(\pi - C)$$

$$= -\tan C$$

Since, the last statement is true, the claim is proved.

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12. $\tan(A+B+C) = \tan[(A+B)+C]$

$$= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - (\tan A \tan B + \tan B \tan C + \tan C \tan A)}$$

If $A+B+C=\pi$, numerator of $\tan(A+B+C) = \tan\pi = 0$

Hence, we get the conclusion of problem 11.

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \tan C$$

$$\Leftrightarrow \frac{\tan A + \tan B + \tan C}{\tan A \times \tan B \times \tan C} = 1$$

$$\Leftrightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1 \quad \text{——— (A)}$$

We have,

$\tan x \tan y = 2$

$\tan y \tan z = 18$

and $x+y+z = \pi$

Find $\tan y$

Using (A), $\frac{1}{2} + \frac{1}{18} + \frac{1}{\tan x \tan z} = 1$

$$\Rightarrow \tan x \tan z = \frac{9}{4}$$

$$\tan y = \tan[\pi - (x+z)]$$

$$= -\tan(x+z)$$

$$\Leftrightarrow -\tan y = \frac{\tan x + \tan z}{1 - \tan x \tan z} = \frac{\tan x \tan y + \tan z \tan y}{\tan y(1 - \tan x \tan z)}$$

$$\Leftrightarrow \quad -\tan^2 y = \frac{2+18}{1-\frac{9}{4}} = \frac{80}{-5}$$

$$\Rightarrow \quad \tan^2 y = 16 \quad \Rightarrow \quad \tan y = \pm 4$$

Suppose $\tan y = -4$, then y is obtuse.
Since $\tan y \tan z > 0$, z is obtuse too.
Since, a Δle cannot have more than one obtuse angle, $\tan y = 4$.

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13. (Bonus problem)

Let $x^3 + ax^2 + bx + c = 0$ have $\cot A$, $\cot B$ and $\cot C$ as roots where A, B, C form the angles of some Δle.

Then, b is ______.

Sol From (A) of problem 12,

$$b = \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$$

Note that,

$$-a = \cot A + \cot B + \cot C$$

$$-c = \cot A \cot B \cot C$$