Problems: week 2

http://bit.ly/trig2013

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Note: *I have given one solution for every problem. There may be other possible solutions. You are encouraged to discuss them. In case you find a error/typo, please mail me.*

1. Let $0 < x < \frac{\pi}{2}$. Prove that $\sin x + \cos x > 1$.

   Sol 1: Let x be an angle in the right angled triangle. Then,

$$\begin{aligned}\sin x + \cos x &= \frac{opp}{hyp} + \frac{adj}{hyp} \\ &= \frac{opp + adj}{hyp} \\ &> 1\end{aligned}$$

   from triangle inequality. (sum of the lengths of any two sides of a triangle is greater than the other side)

2. Prove that

$$\frac{1 - \cos x}{1 + \cos x} = (\csc x - \cot x)^2$$

Sol 1:

$$\begin{aligned}\frac{1-\cos x}{1+\cos x} &= \frac{1-\cos x}{1+\cos x} \times \frac{1-\cos x}{1-\cos x} \quad \text{assuming } 1-\cos x \neq 0 \\ &= \frac{(1-\cos x)^2}{1-\cos^2 x} \\ &= \frac{(1-\cos x)^2}{\sin^2 x} \\ &= (\csc x - \cot x)^2\end{aligned}$$

3. Find 'k' where

$$(\sin x + \csc x)^2 + (\cos x + \sec x)^2 = \cot^2 x + \tan^2 x + k$$

Sol: Assuming LHS is well defined,

$$\begin{aligned}(\sin x + \csc x)^2 + (\cos x + \sec x)^2 &= (\sin^2 x + \csc^2 x + 2\sin x \csc x) + (\cos^2 x + \sec^2 x + 2\sec x \cos x) \\ &= 4 + (\sin^2 x + cos^2 x) + (\sec^2 x + \csc^2 x) \\ &= 5 + (1 + \tan^2 x + 1 + \cot^2 x)\end{aligned}$$

Hence, $k = 7$.

4. If $\sin x + \sin^2 x = 1$, the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x$ is

(a) $\cos^2 x$

(b) 0

(c) 1

(d) none of these

Sol: Since, $\sin x + \sin^2 x = 1$, we have $\sin x = \cos^2 x$.

$$\begin{aligned}
\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x &= \cos^6 x(\cos^6 x + 3\cos^4 x + 3\cos^2 x + 1) \\
&= \cos^6 x(\cos^2 x + 1)^3 \\
&= \sin^3 x(\sin x + 1)^3 \\
&= (\sin x(1 + \sin x))^3 \\
&= (\sin x + \sin^2 x)^3 \\
&= 1 \qquad \text{(using initial condition)}
\end{aligned}$$

5. $\tan 1^\circ \cdot \tan 3^\circ \cdot \tan 4^\circ \cdot \tan 5^\circ \cdots \tan 88^\circ \cdot \tan 89^\circ$ equals to

   (a) $\tan 2^\circ$

   (b) $\cot 2^\circ$

   (c) > 1

   (d) none of these

Sol: We shall prove that $\tan 1^\circ \cdot \tan 2^\circ \cdot \tan 3^\circ \cdot \tan 4^\circ \cdot \tan 5^\circ \cdots \tan 88^\circ \cdot \tan 89^\circ = 1$. Hence the given expression equals to $\cot 2^\circ$. Note that for $0 < x < \frac{\pi}{2}$, $\tan(90^\circ - x^\circ) = \cot x^\circ$. Further, $\tan 45^\circ = 1$. Hence, the product equals to 1.

6. Prove the equality

$$\frac{\tan A - \sec A + 1}{\tan A + \sec A - 1} = \frac{1 - \sin A}{\cos A}$$

Sol: Assuming the LHS is well-defined,

$$\begin{aligned}
\frac{\tan A - \sec A + 1}{\tan A + \sec A - 1} &= \frac{\tan A - \sec A + (\sec^2 A - \tan^2 A)}{\tan A + \sec A - 1} \\
&= \frac{\tan A - \sec A + ((\sec A - \tan A)(\sec A + \tan A))}{\tan A + \sec A - 1} \\
&= \frac{\tan A - \sec A(1 - (\sec A + \tan A)}{\tan A + \sec A - 1} \\
&= \sec A - \tan A \\
&= \frac{1 - \sin A}{\cos A}
\end{aligned}$$

7. $(1+\sin x+\cos x)^2 = 2(1+\cos x)A$. Then , A is

(a) $\cos x$

(b) $1+\sin x$

(c) $\sin x \cos x$

(d) None of these

Sol:

$$\begin{aligned}
(1+\sin x+\cos x)^2 &= (1+(\sin x+\cos x))^2 \\
&= 1+(\sin x+\cos x)^2+2(\sin x+\cos x) \\
&= 1+(\sin^2 x+\cos^2 x+2\sin x\cos x)+2(\sin x+\cos x) \\
&= 2(1+\sin x\cos x+\sin x+\cos x) \\
&= 2(1+\cos x)(1+\sin x)
\end{aligned}$$

8. Prove the equality

$$\frac{\sin^2 x}{1-\cot x}+\frac{\cos^2 x}{1-\tan x} = 1+\sin x\cos x$$

Sol: Assuming that the LHS is well defined,

$$\begin{aligned}
\frac{\sin^2 x}{1-\cot x}+\frac{\cos^2 x}{1-\tan x} &= \frac{\sin^3 x}{\sin x-\cos x}+\frac{\cos^3 x}{\cos x-\sin x} \\
&= \frac{\sin^3 x-\cos^3 x}{\sin x-\cos x} \\
&= \sin^2 x+\cos^2 x+\sin x\cos x \\
&= 1+\sin x\cos x
\end{aligned}$$

9. The value of $2(\sin^6 x+\cos^6 x)-3(\sin^4 x+\cos^4 x)$ equals to

(a) 1

(b) 0

(c) -1

(d) None of these

Sol:

$$\begin{aligned}2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) &= 2\left((\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)\right)\\ &\quad - 3\left((\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x\right)\\ &= 2(1 - 3\sin^2 x\cos^2 x) - 3(1 - 2\sin^2 x\cos^2 x)\\ &= -1\end{aligned}$$

10. If $3\sin x + 5\cos x = 4$. Find the value of $3\cos x - 5\sin x$

Sol: Let $v := 3\cos x - 5\sin x$.

$$\begin{aligned}4^2 + v^2 &= (3\sin x + 5\cos x)^2 + (3\cos x - 5\sin x)^2\\ &= (9\sin^2 x + 25\cos^2 x + 30\sin x\cos x) + (9\cos^2 x + 25\sin^2 x - 30\sin x\cos x)\\ &= 34(\sin^2 x + \cos^2 x)\end{aligned}$$

Hence, $v = \pm\sqrt{18}$

11. Prove that $\cos x + \sin x = \sqrt{2}\cos x \iff \cos x - \sin x = \sqrt{2}\sin x$

Sol:

$$\begin{aligned}\cos x + \sin x &= \sqrt{2}\cos x\\ \sin x &= (\sqrt{2} - 1\cos x)\\ \tan x &= \sqrt{2} - 1\\ &= (\sqrt{2} - 1) \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\\ &= \frac{1}{\sqrt{2}+1}\\ \frac{\sin x}{\cos x} &= \frac{1}{\sqrt{2}+1}\\ \sqrt{2}\sin x + \sin x &= \cos x\\ \cos x - \sin x &= \sqrt{2}\sin x\end{aligned}$$

we could also start from the bottom end at the top.

12. Find the value of $\sin^2 5° + \sin^2 10° + \cdots + \sin^2 90°$

    Sol: Note that for some $0 < x < \frac{\pi}{2}$, $\sin^2 x° + \sin^2(90° - x°) = \sin^2 x° + \cos^2 x° = 1$. Hence, the given sum is $8 + \sin^4 5° + \sin^2 90° = \frac{19}{2}$

13. Let A and B be acute. If $\sin A = 1/2$ and $\cos B = 1/3$. Then, prove that $\frac{\pi}{2} < A+B < \frac{2\pi}{3}$

    Sol: $\sin A = 1/2 \Rightarrow A = \frac{\pi}{6}$. Since cos decreases from 0 to $\pi/2$ and $\cos\frac{\pi}{3} = 1/2$, $\cos\frac{\pi}{3} < B < \cos\frac{\pi}{2}$. Hence, $\frac{\pi}{2} < A + B < \frac{2\pi}{3}$

14. Let $0 < x < \frac{\pi}{4}$. Arrange the following in ascending order.

    $x_1 = (\tan x)^{\tan x}$ $\quad$ $x_2 = (\tan x)^{\cot x}$ $\quad$ $x_3 = (\cot x)^{\tan x}$ $\quad$ $x_4 = (\cot x)^{\cot x}$

    Sol: Note that $0 < \tan x < 1$ and thereby $1 < \cot x$. Hence, x_4 is greater than all the others. Further, $x_2 < x_1$ as $0 < \tan x < 1$ and $1 < \cot x$. $x_1 = (\tan x)^{\tan x} < 1$ while $x_3 = (\cot x)^{\tan x} = \frac{1}{(\tan x)^{\tan x}} > 1$. Hence, we have $x_2 < x_1 < x_3 < x_4$,